Construction of Noether Theory for a Singular Linear Differential Operator.

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Abstract: The main purpose of this work is to construct the noether theory fora singular linear integro-differential operator defined by a first order linear singular differential equation in a specific well choosen functional space. Our various previous published researches were connected with such topic. Case by case, and with respect to the values and sign of the parameter $\gamma \in \mathbb{R}$, we solve the studied differential equation with a specific well known second right hand side $f(x)$,establishing it conditions of solvability, leading us to the investigation and the construction of the noetherity of the operator *L.* Finally, we calculate the deficient numbers and the index of the considered operator in various situations, relatively to the parameter $\gamma \in \mathbb{R}$.

Keywords: noether theory, third kind integral equation, singular linear integro-differential operator, deficient numbers, index of the operator.

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1. Introduction

It is well known that the noether theory for integro-differential operators defined by some integral equations of the third kind is widely studied in some scientific researches (see for example, papers [1,2,4,6,8,10,11,12,23]). One of the main difficulties we face in the case of the investigation of the solvability of integral equation of the third kind while constructing noether theory for operator defined by such integral equations, is to well choose the necessary approaches that lead us to the awaited goal. As previously done in various works, see ([1,2,4,6,7,8,10,13,18]) and illustrated in [22,26], also following Prossdorf. S, Samko S.G.,Kilbas A.A., MarichevO.I, RaslambekovV.S., Gobassov N.S., and Abdourahman, the normalization method, the method of hyper singular integrals and the method of approximative inverses operators were developed for the third kind integral equations in the space of continuous and generalized functions. These methods allowed to set up well the problem when considering integral equations of the first kind, or non Fredholm integral equations of the second kind defining the operators. We also note that Shulaia D.A, and Gugushvili E.I, by their side investigated Inverse problem of spectral analysis of linear multi group neutron transport theory In plane geometry in their paper Transp. Theory Stat.Phys. 29(2000), No. 6, 711-722. Shulaia D.A studied in the class of Holder functions, a nonhomogeneous linear integral equation with coefficient $\cos x$and gave the necessary and sufficient conditions for the solvability of the considered equation undersome assumptions on its kernel. Moreover, he also succeeded and constructed the solution analytically, applying the Fredholm theory and the theory of singular integral equations. For more details see [29]. Analogously, by his side, G. Bal devoted a paper to the Inverse problems for homogeneous transport equations, I. The one-dimensional case which is published. Inverse Problems 16(2000), No. 4, 997-1028.In our previous researches, we have already constructed noether theory for an integro-differential operator defined by a third kind linear singular integral equation of the following form

$$(A\varphi)(x) = x^p \varphi'(x) + \int_{-1}^{1} K(x,t)\,\varphi(t)dt = f(x); x \in [-1,1],$$

with the unknown function $\varphi \in C_{-1}^{1}[-1,1]$, the second right hand side $f(x) \in C_0^{\{p\}}[-1,1]$and the kernel $K(x,t) \in \ C_0^{\{p\}}[-1,1]$X C $[-1,1]$. For full details of such researches, see [11,12, 23, 24, 25]. Going through the same direction, we plan to construct noether theory for an integro-differential operator defined by an integral equation of third kind with a main part of the form of *L,* sothat is why we undertake such investigation as preliminary step. Many researches devoted to third kind integral equations can be found with full details in [2,3,4,5,8,9]. Gobbassov N.S dedicated special investigations on various methods for solving integral equations see [13,14,15,16] and refeer also to [17,18,19,20].

In the present work conducted in the space of continuous functions, we are considering namely a differential operator *L*as main part of the integral operator A, which is defined by an integral equation of the third kind. We focus our investigation only onto the operator*L*for which, we are constructing the noether theory as this has been done in [12, 23], with an important particularity, that in this case, the considered closed interval is [-1,1] rather than [0,1] with the singular interior point zero at this time on the medium of the considered interval.

Namely here, we consider for investigation the linear differential operator:

$$Ly(\mathrm{x}) = xy'(x) + \gamma y(x) = f(x)\,;\ x \in [-1,1]\,, \quad (1)$$

where $\gamma \in \mathbb{R}$ with $f(x) \in C_0^{\{1\}}[-1,1]andy(x) \in C^1[-1,1]$.

The establishment of the noetherity of the operator L is investigated through the solvability of the first order linear differential equation defined by formula (1).The central moment in this work is connected with the construction of the continuity of the regularisators, constructed before in our previous researches when we considered the interval [0,1], to the whole closed interval[-1,1]. See paper [12]. Also for references on such approaches investigated by Yurko V.A with full details, see [30,31]. This lead us to calculate and determine the deficient numbers $\alpha(L)$and $\beta(L)$and also the index $\varkappa(L)$of the operator L, depending of the various cases studied in connexion with the sign and values of γ.

The work is organized as follow: first of all, we present in section 2the preliminaries related to the concept and the notions of well known noether theory. Section 3 is properly devoted to the presentation of the main results of this work and clearly is connected to the cases studied, with dependance of the sign and values of γ, in other words (related to cases a)$\gamma > 0$; b) $\gamma < 0$, $\gamma \neq -1$;c) noetherity for $\gamma \in]-\infty,-1[$;d)$\gamma = 0$). And lastly, we summarize the content of the work in section 4 titled conclusion, followed by some recommendations for the follow-up or future scientific works to undertake, stated in section 5.

2. Preliminaries

Before presenting in details our main results, the following definitions and concepts well known from the noether theory of operators are required. For details in full see also [6,10,12, 21,22].

By the way, we briefly review this important notions of Taylor derivatives, which is widely used when constructing noether theory of the considered operator A.

Let $C^m[-1,1], m \in \mathbb{Z}_+$, noted the Banach space of continuous functions on $[-1,1]$, having continuous derivatives up to order m, for which the norm is defined as following :

$$\|\varphi(x)\|_{C^m[-1,1]} = \sum_{j=0}^{m} \max_{-1\leq x\leq 1}\left|\varphi^{(j)}(x)\right| \qquad (2)$$

So, that we can consider $\varphi^{\{k\}}(0)$ are defined for all $k = 1,2,\dots,p$.

We define $C_0^{\{p\}}[-1,1]$as a subspace of continuous functions, having finite Taylor derivatives up to order $p \in \mathbb{Z}_+$; and when $p = 0$, we put ($C_0^{\{p\}}[-1,1] = C[-1,1]$)).

Let us also define a linear operator N^k on the space $C_0^{\{p\}}[-1,1]$ by the formula:

$$(N^k\varphi)(x) = \frac{\varphi(x) - \sum_{j=0}^{k-1}\frac{\varphi^{\{j\}}(0)}{j!}x^j}{x^k}, k = 1,2,\dots,p, \qquad (4)$$

One can easily verify the property $N^k = N^{k_1}N^{k-k_1}, 0 \leq k_1 \leq k, k, k_1 \in \mathbb{Z}_+$, where we put $N^0 = I$. The operator N^p characterizes the space $C_0^{\{p\}}[-1,1]$ as it can be seen from the following lemma see also [23, 28, 29].

Lemma 2.1. A function $\varphi(x)$belongs to$C_0^{\{p\}}[-1,1]$ if and only if, the following representation

$$\varphi(x) = x^p\phi(x) + \sum_{k=0}^{p-1}\alpha_k\, x^k, \qquad (5)$$

holds with $\phi(x) \in C[-1,1]$, and α_k being constants.

To prove Lemma 2.1 it is enough to observe that (5) implies that the Taylor derivatives of $\varphi(x)$ up to the order $p$exists, and $\varphi^{\{k\}}(0) = k!\,\alpha_k, k = 0,1,2,\dots,p-1, \varphi^{\{0\}}(0) = p!\,\phi(0)$ with $\phi(x) = (N^k\varphi)(x)$. Conversly, if $\varphi(x)$belongs to$C_0^{\{p\}}[-1,1]$, and we define $\phi(x) = (N^k\varphi)(x)$ with $\alpha_k = \frac{\varphi^{\{k\}}(0)}{k!}, k = 0,1,2,\dots,p-1$,then the representation (5) holds. From Lemma 2.1, it follows that for $\varphi(x) \in C_0^{\{p\}}[-1,1]$ the inequality

$$\varphi(x) = x^p(N^k\varphi)(x) + \sum_{k=0}^{p-1}\frac{\varphi^{\{k\}}(0)}{k!}x^k, \qquad (6)$$

is valid.

Consequently, the linear operator N^p establishes a relation between the spaces $C_0^{\{p\}}[-1,1]$ and $C[-1,1]$. The space $C_0^{\{p\}}[-1,1]$ with the norm

$$\|\varphi\|_{C_0^{\{p\}}[-1,1]} = \|N^p\varphi\|_{C[-1,1]} + \sum_{k=0}^{p-1}\left|\varphi^{\{k\}}(0)\right|, \qquad (7)$$

becomes a Banach space one.

Note that, it is obvious to see that $\|\varphi\|_{C[-1,1]} \leq$ φ *C0p−1,1.*

Finally, note that from the definition 2.1 it follows the following fact, if $\varphi(x) \in C[-1,1]$, then $x^p\varphi(x) \in C_0^{\{p\}}[-1,1]$. This assertion may be generalized as follows.

Lemma 2.2. Let $p \in \mathbb{N}, s \in \mathbb{Z}_+$. If $\varphi(x) \in C_0^{\{s\}}[-1,1]$ then, $x^p\varphi(x) \in C_0^{\{p+s\}}[-1,1]$, and the formula holds

$$\left(x^p\varphi(x)\right)^{\{j\}}(0) = \begin{cases} 0, \ j = 0,1,\dots,p-1, \\ \frac{j!}{(j-p)!}\varphi^{\{j-p\}}(0), j = p,\dots,p+s. \end{cases} \qquad (8)$$

Proof. Note that a stronger assumption on the function

$\varphi(x)$, such that $\varphi(x) \in C_0^{\{p+s\}}[-1,1]$ would allow us to easily prove the lemma just by applying Leibniz formula.

For $s = 0$ the statement has already been proved above, so $x^p\varphi(x) \in C_0^{\{p\}}[-1,1]$,and $\left(x^p\varphi(x)\right)^{\{j\}}(0) = 0, j = 0,\dots,p-1$ and $\left(x^p\varphi(x)\right)^{\{p\}}(0) = p!\,\varphi(0)$. Now let us prove that $\left(x^p\varphi(x)\right)^{\{j\}}(0) = \frac{j!}{(j-p)!}\varphi^{\{j-p\}}(0), j = p+1,\dots.,p+s$. Since the derivatives are defined reccurently, and (8) is true for $j = p$, then it is sufficient to verify the passage from j to $j+1$. We have:

$$\left(x^p\varphi(x)\right)^{\{j+1\}}(0) =$$

$$(j+1)!\lim_{x\to 0}\frac{x^p\varphi(x)-\sum_{l=p}^{j}\frac{x^l}{(l-p)!}\varphi^{\{l-p\}}(0)}{x^{j+1}} \quad (9)$$

$$= (j+1)!\lim_{x\to 0}\frac{\varphi(x)-\sum_{l=0}^{j-p}\frac{x^l\varphi^{\{l\}}(0)}{l!}}{x^{j+1-p}} = \frac{(j+1)!}{(j+1-p)!}\varphi^{\{j+1-p\}}(0).$$

(10)

Lemmas 2.1 and 2.2 imply the next lemma.

Lemma 2.3. Let $f(x) \in C_0^{\{p\}}[-1,1], p \in \mathbb{N}$ *and* $f(0) =$ *.....=fr−10=0, 1≤r≤p.*Then *f(x)xr∈C0p−s−1,1.*

It is also convenient to use an equivalent definition for the norm in $C_0^{\{p\}}[-1,1]$:

$$\|\varphi\|^1_{C_0^{\{p\}}[-1,1]} = \sum_{j=0}^{p}\left\|N^j\varphi\right\|_{C[-1,1]} \quad (11)$$

It is easy to verify the equivalence of the norms (7) and (11). Namely, we always have $\left|\varphi^{\{j\}}(0)\right| \le j!\left\|N^j\varphi\right\|_{C[-1,1]}$ which gives the estimate $\|\varphi\| \le c\|\varphi\|^1$. To obtain the inverse estimate, we use the next following equality below.

$$(N^j\varphi)(t) = t^{p-j}(N\varphi)(t) + \sum_{s=j}^{p-1}\frac{t^{s-j}}{s!}\varphi^{\{s\}}(0), \quad (12)$$

from which $\left\|N^j\varphi\right\|_{C[-1,1]} \le \|\varphi\|, j = 0,1,\dots..,p-1$ and, then $\|\varphi\|^1 \le c\|\varphi\|$.

Lemma 2.4. The operator $N^p: C_0^{\{p\}}[-1,1] \to C[-1,1]$ has the following properties:

1. N^p is bounded, and $\|N^p\varphi\|_{C[-1,1]} \le \|\varphi\|_{C_0^{\{p\}}[-1,1]}$;

2. N^p is right invertible;

3. $\alpha(N^p) = p$, where $\alpha(N^p)$ is the dimension of the null subspace for N^p.

Proof. Statement 1) follows from the definition of the norm in (7). The invertibility is justified by the fact that the equation $N^p\varphi = f$ with an arbitrary $f(x) \in C[-1,1]$ has a solution $\varphi(x) = x^p f(x) \in C_0^{\{p\}}[-1,1]$, which follows from Lemma 2.1 and the equality $N^p(x^p f(x)) \equiv f(x)$. By 2) and noticing that $N^p x^k = 0$ for all $k = 0,1,\dots.,p-1$, we arrive at to the point 3).

The lemma is proved.

Note that from the proof of this lemma, it follows that the equation $N^p f = g$ is always solvable in the space $C_0^{\{p\}}[-1,1]$ for every $g(x) \in C[-1,1]$, and its general solution has the form

$$f(x) = \sum_{k=0}^{p-1} c_k\, x^k + x^p g(x), \quad (13)$$

where c_k are arbitrary constants. Note that for the Taylor derivatives, many formulas are valid similar to those for ordinary derivatives; in particular, Leibniz formula, l'Hospitale rule and others see [13]. Nevertheless, for example, the formula

$$\left|\varphi^{(k)}(x)\right|^{\{r\}}(0) = \varphi^{\{k+r\}}(0) \quad (14)$$

is not always valid.

Next, let us move to the presentation of the general important results of the work in the following section.

3. Main Results

In this section we undertake properly the noetherity investigation of the operator L.

Namely, here we consider as model to be investigated, the differential equation of the following type:

$$Ly(\mathrm{x}) = xy'(x) + \gamma y(x) = f(x)\,;\ x \in [-1,1]$$

where $\gamma \in \mathbb{R}$and $y(x) \in C^1[-1,1]$ with $f(x) \in$ *C01−1,1.*

A. The case $\gamma > 0$.

Recall that when $x \in [0,1]$, this case has been studied in [12, 23] and the solution $y(x)$ was given by the formula:

$$y(x) = x^{-\gamma} \int_0^x t^\gamma (Nf)(t)dt + \frac{f(0)}{\gamma}, x > 0 \quad (15)$$

Accordingly to the form of the particular solution of the equation (1), when $x < 0$, and $f(x) \in C_0^{\{1\}}[-1,1]$, then by direct verification we should take it as following:

$$y(x) = -\int_x^0 \left(\frac{|t|}{|x|}\right)^\gamma (Nf)(t)dt + \frac{f(0)}{\gamma}, x < 0 \quad (16)$$

with respect to $\gamma y(0) = f(0)$. We can therefore formaly write the general form of the particular solution of the non homogeneous equation (1) when $\gamma > 0$.

$$y(x) = \begin{cases} \int_0^x \left(\frac{t}{x}\right)^\gamma (Nf)(t)dt + \frac{f(0)}{\gamma}, x > 0, \\ \frac{f(0)}{\gamma}, \quad x = 0 \\ -\int_x^0 \left(\frac{|t|}{|x|}\right)^\gamma (Nf)(t)dt + \frac{f(0)}{\gamma}, x < 0. \end{cases} \quad (17)$$

Of course, it is necessary to verify that the expression (17) is solution of the equation (1) and moreover $y(x) \in C^1[-1,1]$.

For this matter we need the following important lemma.

Lemma 3.1. Let the kernel $k(x,t)$ be homogeneous kernel of degree -1 and satisfies the integrability condition $\int_0^x |k(-1,-t)|dt < \infty$. Then for $g(t) \in C\,]-\infty, 0]$ it holds the relation

$$\lim_{x \to -0} \int_{-\infty}^0 k(x,t)\ g(t)dt = g(-0) \int_0^\infty k(-1,-t)dt \quad (18)$$

For the proof let us use theorem 5.1 from [23]. Setting $y \longrightarrow -t$ and further $t = -xy$, with consideration to $x < 0$ we have the equality $\int_{-\infty}^0 k(x,y)g(y)dy =$ *0∞k(−1,−y)g(xy)dy,* from where and by the same theorem 5.1 it follows (18).

Now, we verify that for the solution $y(x)$ the following relations hold.

$$\begin{cases} \lim_{x\to -0} y(x) = \lim_{x \to +0} y(x) = y(0) = \frac{f(0)}{\gamma}, \\ \lim_{x\to -0} y'(x) = \lim_{x \to +0} y'(x) = y'(0). \end{cases} \quad (19)$$

Let us begin with the first relation in (19). Setting $g(t) = t(Nf)(t)$, it is easy to see that the integral operator in (17) with the kernel $k(x,t) = |x|^{-\gamma}|t|^{-\gamma-1}$satisfies the integrability condition using once again theorem 5.1 from [23]when $x > 0$,and lemma 3.1 when $x < 0$and considering that $g(\pm 0) = 0$, we arrive to (19).

Finally, we establish the validity of the second relation in (19) when $x < 0$.

$$y'(x) = (Nf)(x) - \gamma|x|^{-1} \int_x^0 \left(\frac{|t|}{|x|}\right)^\gamma (Nf)(t)dt \quad (20)$$

Here, the kernel of the integral operator in (20) $k(x,t) = -\gamma|t|^{-\gamma}|x|^{-\gamma-1}$ satisfies the integrability condition from Lemma 3.1 and is homogeneous of degree -1. Therefore, the Lemma 3.1 on the limit passage may be applied. With consideration to what has been said for the calculation of $\lim_{x\to -0} y'(x)$, we have the following relation.

$$\lim_{x\to -0} y'(x) =$$
$$\lim_{x\to -0}(Nf)(x) - \gamma \lim_{x \to -0} (Nf)(x) \int_0^1 t^\gamma dt = \frac{f(0)^{\{1\}}}{\gamma+1} (21).$$

Analogously, but by now using theorem 5.1 from [23],

we find that $\lim_{x\to +0} y'(x) = (\gamma+1)^{-1} f(0)^{\{1\}}$. It remains to

calculate the value $y'(0)$. By the definition of the derivative,

we have: $y'(0) = \lim_{x\to 0} \omega(x)$, where

$$\omega(x) = \begin{cases} x^{-1} \int_0^x \left(\frac{t}{x}\right)^\gamma (Nf)(t)dt, x \ge 0, \\ |x|^{-1} \int_x^0 \left(\frac{|t|}{|x|}\right)^\gamma (Nf)(t)dt, x < 0. \end{cases} \quad (22)$$

and moreover, it is easy to show that:

$$\lim_{x\to 0} \omega(x) = \lim_{x \to \pm 0} \omega(x) = (\gamma+1)^{-1} f(0)^{\{1\}}.$$

From the previous ideas it follows directly the following theorem.

Theorem 3.1. Let $\gamma > 0$ and $f(x) \in {C_0}^{\{1\}}[-1,1]$. Then the equation (1) is solvable for any right hand side $f(x)$ and has a unique solution from $C^1[-1,1]$, defined by the formula (17).

<u>Proof</u>. First of all we give an accurate operator interpretation. For this matter let us introduce the following notation. Let the operators L and L^{-1} defined as following $L: C^1[-1,1] \longrightarrow {C_0}^{\{1\}}[-1,1]$and $L^{-1}: {C_0}^{\{1\}}[-1,1] \longrightarrow C^1[-1,1]$, where the operators Ly defined by (1) and $L^{-1}f$ expressed by the right hand side in (17).

$$(L^{-1}f)(x) = \begin{cases} \int_0^x \left(\frac{t}{x}\right)^\gamma (Nf)(t)dt + \frac{f(0)}{\gamma}, x < 0, \\ \frac{f(0)}{\gamma}, x = 0, \\ -\int_x^0 \left(\frac{|t|}{|x|}\right)^\gamma (Nf)(t)dt + \frac{f(0)}{\gamma}, x > 0. \end{cases} \quad (23)$$

But before, remark that the operators L and L^{-1} are bounded. Indeed, by the definition of the Taylor derivative with respect to $(Ly)(0) = 0$ we have:

$$(Ly)^{\{1\}}(0) = \lim_{x\to 0}\left[y'(x) + \gamma\frac{y(x)-y(0)}{x}\right] = (1+\gamma)y'(0) \quad (24)$$

and that is why

$$\|Ly\|_{{C_0}^{\{1\}}[-1,1]} = \|Ly\|_{C[-1,1]} + \|NLy\|_{C[-1,1]} \leq \mathrm{C}\|y\|_{C^1[-1,1]} \quad (25)$$

that gives the boundedness of the operator L.

Now let us turn to the operator L^{-1} and show, that it is bounded. Simple approximations and calculations give the following result:

$$\|L^{-1}f\|_{C^1[-1,1]} = \max_{-1\leq x\leq 1}|(L^{-1}f)(x)| + \max_{-1\leq x\leq 1}\left|\frac{d}{dx}(L^{-1}f)(x)\right| \leq$$
3γ+1γ+1(Nf)(x)C−1,1+1γf(x)C−1,1.

(26)

So that the operator $L^{-1}f$ is bounded.

Further we calculate $LL^{-1}f$ with the function $f(x) \in {C_0}^{\{1\}}[-1,1]$. When $x > 0$, we have showed that $LL^{-1}f = f$. See full details also in [23].

Analogously we have when $x < 0$

$$LL^{-1}f = x\left(-\int_x^0 \left(\frac{|t|}{|x|}\right)^\gamma (Nf)(t)dt + \frac{f(0)}{\gamma}\right)' + \gamma\frac{f(0)}{\gamma} + \gamma\int_x^0 \left(\frac{|t|}{|x|}\right)^\gamma (Nf)(t)dt = f(x). \quad (27)$$

So for all $x \in [-1,1]$,we get the relation $LL^{-1}f = f, f(x) \in {C_0}^{\{1\}}[-1,1]$, and L^{-1} is right invertible bounded operator. Now let us consider the action of the operator $L^{-1}Ly$ onto the functions $y(x) \in C^1[-1,1]$. As we know $(L^{-1}Ly)(x) = y(x)$ when $x > 0$.

In the case $x < 0$ with respect to the relations $(Ly)(0) = \gamma y(0)$,

$(NLy)(x) = y'(x) + \gamma(Ny)(x)$ and simple calculations connected with the integration by parts of the integral containing $y'(t)$, we have the following:

$$(L^{-1}Ly)(x) = -\int_x^0 \left(\frac{|t|}{|x|}\right)^\gamma (NLy)(t)dt + \frac{(Ly)(0)}{\gamma} =$$
$$= -\int_x^0 \left(\frac{|t|}{|x|}\right)^\gamma (y'(t)dt + \gamma(Ny)(t))dt + y(0)$$
$$= y(x). \quad (28)$$

So that $(L^{-1}Ly)(x) = y(x)$ for all $x \in [-1,1]$. The theorem is proved.

Remark that in this case, we have a natural setting of the problem from $C^1[-1,1]$ into ${C_0}^{\{1\}}[-1,1]$ and, moreover the differential operator defined by the left hand side in (1) is invertible.

Let us move to the following situation.

B. The case $\gamma < 0, \gamma \neq -1$.

Here contrary to the case when $\gamma > 0$, the homogeneous equation (1) may have a solution in the form of

$y_0(x) = c_1|x|^{-\gamma} + c_2|x|^{-\gamma} signx$ (29).

If $-\infty < \gamma < -1$. In the contrary case, when $\gamma \in (-1,0)$ in the formula (29) we have to set $c_j = 0, j = 1,2$.When constructing particular solution of the non homogeneous equation (1) it follows to take in consideration that when $\gamma < -1$, the functions in (29) generate a non integrable singularity at the point zero and, this influence on the choise of the form of the solutions. It takes place the following.

Lemma 3.2. Let $\gamma < -1$ and $(Nf)(t) \in C[-1,1]$. Then the solution of the equation (1) in $C^1[-1,1]$ is not unique and is defined by the following formula:

$$y(x) = c_1|x|^{-\gamma} + c_2|x|^{-\gamma} signx +$$

$$+\begin{cases} -\int_x^1 \left(\frac{t}{x}\right)^{\gamma} (Nf)(t)dt + \frac{f(0)}{\gamma}(1 - x^{-\gamma}), x > 0, \\ \frac{f(0)}{\gamma}, x = 0, \\ \int_{-1}^x \left(\frac{|t|}{|x|}\right)^{\gamma} (Nf)(t)dt + \frac{f(0)}{\gamma}(1 - |x|^{-\gamma}), x < 0. \end{cases} \quad (30)$$

where c_j $(j = 1,2)$ are arbitrary constants.

Proof. First of all remark that the case when $\gamma \in (-1,0)$ can be included into the lemma, setting $c_j = 0, j = 1,2$ in the formula of the solution. Now verify separately that every expression of the particular solution $y_{par}(x)$ when $(c_1 = c_2 = 0)$, is solution of the equation (1). The fact that it is true when $x > 0$ follows from the previous results obtained in [23].

Now if $x < 0$, we have

$y'(x) = \gamma(-x)^{-\gamma-1} \int_{-1}^x |t|^{\gamma} (Nf)(t)dt + (Nf)(x) +$ *f(0)−x−γ−1* and when substituting into the equation (1) we have $xy' + \gamma y = f(x)$.

Analogously, as in the case when $\gamma > 0$, we verify that it takes place the sewing of the solutions i.e the relation (19)hold. For this goal, we need the following lemmas which are consequence of theorem 5.1 see [23] and, Lemma 3.1.

Lemma 3.3. Let $k(x,y)$ homogeneous of degree -1 and satisfies the integrability condition $\int_1^{\infty} |k(1,t)|dt < \infty$. Then for any function $f(y) \in C[0,1]$ it holds the relation:

$$\lim_{x\to+0} \int_x^1 k(x,y)f(y)\,dy = f(+0)\int_1^{\infty} k(1,t)dt \quad (31)$$

Lemma 3.4. Let $k(x,y)$ homogeneous of degree -1 and satisfies the integrability condition $\int_1^{\infty} |k(-1,-t)|dt < \infty$, then for any function $f(y) \in C[-1,0]$ it holds the relation:

$$\lim_{x\to-0} \int_{-1}^x k(x,y)f(y)\,dy = f(-0)\int_1^{+\infty} k(-1,-t)dt. (32)$$

Now on the basis of these lemmas, we show the sewing of the particular solutions. For this goal, we calculate firstly:

$$\lim_{x\to+0} y(x) = \frac{f(0)}{\gamma} - \lim_{x\to+0} \int_x^1 \left(\frac{t}{x}\right)^{\gamma} (Nf)(t)dt. (33)$$

Remarking that here the kernel $k(x,t) = t^{\gamma}x^{-\gamma-1}$ is homogeneous of degree -1 and also with respect to $\gamma < -1$ satisfies the integrability condition and $(Nf)(t) \in C[0,1]$, we get $\lim_{x\to+0} y(x) = \gamma^{-1}f(0)$. Analogously when $x \to -0$, we have:

$$\lim_{x\to-0} y(x) = \frac{f(0)}{\gamma} + \lim_{x\to-0} \int_{-1}^x \left(\frac{|t|}{|x|}\right)^{\gamma} (Nf)(t)dt =$$
f(0)γ.(34)

By the same way, we verify that the left derivative and the right derivative of the solution $y(x)$ are equal. We have:

$$\lim_{x\to+0} y'(x) = -\lim_{x\to+0} \int_x^1 \frac{1}{x}\left(\frac{t}{x}\right)^{\gamma} (Nf)(t)dt = \frac{f^{\{1\}}(+0)}{\gamma+1}, (35)$$

where , once again we apply the Lemma 3.3 to the kernel $k(x,t) = t^{\gamma}x^{-\gamma-1}$, $\gamma < -1$ and took into account that $(Nf)(0) = f^{\{1\}}(+0)$. In the order side, when $x \to 0$ by the left side we have:

$$\lim_{x\to-0} y'(x) = -\lim_{x\to-0} \int_{-1}^x \left(-\frac{1}{x}\right)\left(\frac{|t|}{|x|}\right)^{\gamma} (Nf)(t)dt =$$
f1−0γ+1.(36)

It remains to calculate the value of the derivative into the point $x = 0$. By the definition we have:

$$y'(0) = \lim_{\substack{x\to0 \\ x\neq0}} \frac{y(x)-y(0)}{x} = \frac{f^{\{1\}}(0)}{\gamma+1} (37)$$

independantly to the way that x tend to zero. With consideration to the previous ideas and assumptions that $f(x) \in C_0^{\{1\}}[-1,1]$, we conclude that $\lim_{x\to+0} y'(x) =$

$\lim_{x\to -0} y'(x) = y'(0) = \frac{f^{\{1\}}(0)}{\gamma+1}$, so that the second relation in (18)is realized. Thus, we have verified that the solution $y(x)$ defined by (30)belongs to the class $C^1[-1,1]$.

Let us combine all what has been said into the solvability theorem of the equation (1).

Theorem 3.2. Let$\gamma < 0, \gamma \neq -1$ and $f(x) \in C_0{}^{\{1\}}[-1,1]$. Then the equation (1) is solvable for any right hand side $f(x)$ and has a unique solution $y(x)$ from the class $C^1[-1,1]$ defined by the formula (30) in the case when $\gamma \in]-\infty,-1[$ and when $-1 < \gamma < 0$ in the formula (30) we set $c_j = 0, j = 1,2$.

Remark that here we have a natural setting of the problem from the class $C^1[-1,1]$ into $C_0{}^{\{1\}}[-1,1]$ and moreover, the differential operator defined in the left hand side in (1) is noether with the index $\varkappa(L) = 2$ and deficient numbers $(\alpha,\beta) = (2,0)$ in the case $\gamma \in]-\infty,-1[$ and when $\gamma \in (-1,0)$ the operator will be invertible.

Now let us move to the following situation:

C. Noetherity in the case $\gamma \in]-\infty,-1[$

Let us introduce the following notations. Let the operators L and L^{-1} be defined as follows: $L: C^1[-1,1] \longrightarrow C_0{}^{\{1\}}[-1,1]$ acting as previous and defined by the formula (1) and

$L^{-1}: C_0{}^{\{1\}}[-1,1] \longrightarrow C^1[-1,1]$ has the form

$$(L^{-1}f)(x) = \begin{cases} \int_{-1}^{x} \left(\frac{|t|}{|x|}\right)^{\gamma} (Nf)(t)dt + \frac{f(0)}{\gamma}(1-|x|^{-\gamma}), x<0, \\ \frac{f(0)}{\gamma}, x=0, \\ -\int_{x}^{1} \left(\frac{t}{x}\right)^{\gamma} (Nf)(t)dt + \frac{f(0)}{\gamma}(1-x^{-\gamma}), x>0, \end{cases} \quad (38)$$

Remark that here we can take $\gamma \in]-\infty,-1[\cup (-1,0)$. Now show that the operators L and L^{-1} are bounded. Indeed, for the operator L the ideas as in the case $\gamma > 0$, remain so that $\|Ly\|_{C_0{}^{\{1\}}[-1,1]} \leq \mathrm{c}\|y\|_{C^1[-1,1]}$.

Concerning L^{-1}, we have the following situation:

$$\|L^{-1}f\|_{C^1[-1,1]} = \max_{-1\leq x\leq 1}|L^{-1}f| + \max_{-1\leq x\leq 1}\left|\frac{\mathrm{d}}{\mathrm{dx}}L^{-1}f\right| \leq \max\Big[\max_{-1\leq x\leq 0}\Big|(1-|x|^{-\gamma})\frac{f(0)}{\gamma} + \int_{-1}^{x}\left(\frac{|t|}{|x|}\right)^{\gamma}(Nf)(t)dt\Big|, \max_{0\leq x\leq 1}\Big|-\int_{x}^{1}|t|^{\gamma}(Nf)(t)dt + \frac{f(0)}{\gamma}(1-x^{-\gamma})\Big|\Big] +$$
$$\max\Big[\max_{-1\leq x\leq 0}\Big|\gamma(-x)^{-\gamma-1}\int_{-1}^{x}\left(\frac{|t|}{|x|}\right)^{\gamma}(Nf)(t)dt + (Nf)(x) + (-x)^{-\gamma-1}f(0)\Big|, \max_{0\leq x\leq 1}\Big|\gamma(x)^{-\gamma-1}\int_{x}^{1}t^{\gamma}(Nf)(t)dt + (Nf)(x) + (x)^{-\gamma-1}f(0)\Big|\Big], \quad (39)$$

from where with respect to $\gamma < -1$ and lemmas 3.3 and 3.4 we can easily deduce the estimate $\|L^{-1}f\|_{C^1[-1,1]} \leq \|f\|_{C_0{}^{\{1\}}[-1,1]}$, what was claimed.

Our next goal is to find the values of the corresponding operators composition L and L^{-1} in the corresponding spaces. Namely, calculate firstly $L\,L^{-1}f = x(L^{-1}f)' + \gamma L^{-1}f$. When $x \geq 0$, we have already verified that $(L\,L^{-1}f)(x) = f(x)$.

Analogously we can verify that the last equality is also valid when $x \leq 0$. Therefore $L\,L^{-1}f = f$ and, consequently the operator L^{-1} is bounded and right invertible operator.

Further, we consider the value of the composition in a different order making calculations with respect to the sign of x. We begin with the case when $x > 0$, we have:

$$(L^{-1}Ly)(x) = y(0)(1-x^{-\gamma}) - x^{-\gamma}\int_{x}^{1}t^{\gamma}\Big[y'(t) + \gamma y(t) - \gamma y(0)tdt. \quad (40)$$

Integrating by parts in the first integral in the right side and making simple calculations, we easily get $(L^{-1}Ly)(x) = y(x) - x^{-\gamma}y(1)$. Similar ideas in the case $x < 0$ give the following result:

$$L^{-1}Ly = \int_{-1}^{x}\left(\frac{|t|}{|x|}\right)^{\gamma}(NLy)(t)dt + \frac{Ly(0)}{\gamma}(1-|x|^{-\gamma}) =$$
y(0)(1−x−γ)+x−γ−1x(−t)γy′(t)+γy(t)−γy(0)tdt=
y(x)−y(−1)x−γ.
(41)

Therefore, when $x < 0$ we get :

$$(L^{-1}Ly)(x) = y(x) - y(-1)|x|^{-\gamma}. \quad (42)$$

So, for all x:

$$(L^{-1}Ly)(x) = (I - T_1)y, (T_1y)(x) = \begin{cases} x^{-\gamma}y(1), x \geq 0, \\ |x|^{-\gamma}y(-1), x \leq 0. \end{cases} \quad (43)$$

It is clear that T_1 is a two dimensional operator satisfying the following condition: $({T_1}^2 y)(x) = (T_1 y)(x)$, so T_1 is a projection. Thus the operator L is a noether operator with the index $\varkappa(L) = 2$ and deficient numbers $(\alpha, \beta) = (2,0)$.

Finally, let us move to the last following situation.

D.The case $\gamma = 0$.

This case is very simple. The equation (1) correspond to the equation $xy'(x) = f(x)$ which is solvable in the class $C^1[-1,1]$ when $f(0) = 0$ and has a solution given by the formula:

$$\int_{-1}^{x} (Nf)(t)dt + c, \tag{44}$$

where c is an arbitary constant. If we introduce the solutions: $(Ly)(x) = xy'(x)$ and $L^{-1}f = \int_{-1}^{x} (Nf)(t)dt$, then it is very easy to see that the operators $L: C^1[-1,1] \longrightarrow {C_0}^{\{1\}}[-1,1]$ and $L^{-1}: {C_0}^{\{1\}}[-1,1] \longrightarrow C^1[-1,1]$, are bounded and the following relations are valid:

$$(L\, L^{-1}f)(x) = (I - T_1)f, \text{ where } (T_1 f)(x) = f(0) \tag{45}$$

and

$$(L^{-1}Ly)(x) = (I - T_2)y\,, \text{ where } (T_2 y)(x) = y(-1). \tag{46}$$

Moreover, it is not difficult to verify that the operators T_1 and T_2 are projections.

Lemma 3.5 Let $\gamma = 0$ and$f(x) \in {C_0}^{\{1\}}[-1,1]$. Then the equation (1) is solvable for any right side $f(x)$ under the accomplished of the solvability condition $f(0) = 0$ and has the solution $y(x)$ from the class $C^1[-1,1]$,defined by the formula operator (44). The operator L, defined (when $\gamma = 0$) by the left side in the equation (1) and acting from $C^1[-1,1]$ into ${C_0}^{\{1\}}[-1,1]$, is a noether operator with the index $\varkappa(L) = 0$ and deficient numbers $(\alpha, \beta) = (1,1)$.

Finally, let us move to the conclusion of the work in the following next section.

4. Conclusion

This achieved scientifc work presents in full details the completed investigation of the construction of noether theory for the operator L, defined by a first order linear singular differential equation in the space of continuous functions $C^1[-1,1]$.

We have found the solvability condition of the equation (1) taking into account the nature of the parameter $\gamma \in \mathbb{R}$ in various cases. This lead us to determine the deficient numbers $(\alpha, \beta)$and therefore the index $\varkappa(L)$ of the operator L, which is finite in all cases, making the operator to be noether.

5. Recommendations

It is obvious to naturally note that, we have realized the most important investigation necessary to undertake the study for noetherity question of an integro-differential operator, defined by a third kind singular integral equation which, has the considered studied operator L as main part. We know from the general theory that, under perturbation of a noether operator by a compact operator and, in the case to be investigated in a brief future, we will reach and maintain the noetherity nature of the initial oparator L. This will be the next future work to undertake when we do consider first of all, the operator A as a sumof two operators L and K where,L is the operator defined by $Ly(\mathrm{x}) = xy'(x) + \gamma y(x) = f(x)$ and$K$ is a compact operator defined as follows $K\varphi = \int_{-1}^{1} k(x,t)y(t)dt$.

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