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Noetherization of a Singular Linear Differential Operator

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Abstract: *The main purpose of this work is to realize the noetherity construction theory fora singular linear differential operator L defined by a first order linear singular differential equation in a specific well-chosen functional space. We have investigated such problems related with this topic in our various previous published researches in the case when* $p \in \mathbb{N}, p = 1$. *Step by step and and with respect to the various cases investigated connected with the values and sign of the parameters* $p \in \mathbb{N}, p \geq 2$ *and* $\gamma \in \mathbb{R}$, *we solve the studied differential equation with a specific well known second right hand side* $f(x)$. *The establishment of it conditions of solvability, lead us to the investigation and the construction of the noetherity of the considered operator L. We therefore finally compute the deficient numbers and the index of the considered operator in various situations, relatively to the parameters* $p \in \mathbb{N}, p \geq 2$ *and* $\gamma \in \mathbb{R}$.

Keywords and phrases: *noether theory, third kind integral equation, singular linear integro-differential operator, deficient numbers, index of the operator.*

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1. Introduction

The construction of noether theory for some integro-differential operators defined by some integral equations of the third kind widely studied in some scientific researches (see for example, papers [1,2,4,6,8,10,11,12,23]) is well known process dealing with serious results from differential equations, functional analysis and integral equations theories. However, due to some specifities and, expressed as one of the main difficulties related to such investigation, we face sometimes in this case the problem of chosing the indicated approach to reach the goal. Recall that the investigation of the solvability of integral equation of the third kind while constructing noether theory for operator defined by such integral equations lead us to well choose the necessary approaches which conduct us to the awaited goal. Various published works, see ([1,2,4,6,7,8,10,13,18]) and illustrated in [22,26], also following the researches of Prossdorf. S, Samko S.G., Kilbas A.A., Marichev O.I, Raslambekov V.S., Gobassov N.S., and Abdourahman, it has been mentionned for example the normalization method, the method of hypersingular integrals. We do not for get an important approach called method of approximative inverses operators which were developed for the third kind integral equations in the space of continuous and generalized functions, clearly illustrated in the researches of many scientits. These cited and mentionnned methods permitted us to set up well the problem when considering integral equations of the first kind, or non Fredholm integral equations of the second kind defining the operators. We also note that Shulaia D.A, and Gugushvili E.I, by their side investigated Inverse problem of spectral analysis of linear multigroup neutron transport theory In plane geometry in their paper Transp. Theory Stat.Phys. 29(2000), No. 6, 711-722. Shulaia D.A studied in the class of Holder functions, a nonhomogeneous linear integral equation with coefficient $\cos x$and gave the necessary and sufficient conditions for the solvability of the considered equation under some assumptions on its kernel. Moreover, he also succeeded and constructed the solution analytically, applying the Fredholm theory and the theory of singular integral equations. For more details see [29].

Analogously, by his side, G. Bal devoted a paper to the Inverse problems for homogeneous transport equations, I. The one-dimensional case which is published. Inverse Problems 16(2000), No. 4, 997-1028. Our previous published researches, clearly indicated the noetherity construction for an integro-differential operator defined by a third kind linear singular integral equation of the following form

$$(A\varphi)(x) = x^p \varphi'(x) + \int_{-1}^{1} K(x,t)\,\varphi(t)dt = f(x); x \in [-1,1],$$

with the unknown function $\varphi \in C_{-1}^{1}[-1,1]$, the second right hand side $f(x) \in C_0^{\{p\}}[-1,1]$ and the kernel $K(x,t) \in C_0^{\{p\}}[-1,1]$X C $[-1,1]$. Details of such researches in full can be found, see [11,12, 23, 24, 25]. Our recent publication [32]illustrated completely an approach undertook to reach the goal of the noetherity construction of an operator *L,* being the main part of the integro-differential operator Adefined by a third kind linear singular integral equation of the following form:

$$(Ay)(x) = xy'(x) + \gamma y(x) + \int_{-1}^{1} K(x,t)\,y(t)dt = f(x); x \in [-1,1],$$

with the unknown function $y \in C_{-1}^{1}[-1,1]$. Relatively and with respect to the values and sign of the parameter $\gamma \in \mathbb{R}$, the noetherity construction of the operator L has been achieved completely and it defient numbers with the index have been calculated, making the operator L a noether one. Developping such investigations and going through the same direction, we plan in this paper analogously to construct noether theory for an integro-differential operator defined by an integral equation of third kind with a main part of the form of *L,* with the same important particularity, that in this case, the considered closed interval is [-1,1] rather than [0,1] with the singular interior point zero at this time on the medium of the considered interval.

Namely here, we consider for investigation the linear differential operator:

$$Ly(x) = x^p y'(x) + \gamma y(x) = f(x)\,;\; x \in [-1,1]\,, \qquad (1)$$

where $p \geq 2$, $\gamma \in \mathbb{R}$ with $f(x) \in C_0^{\{p\}}[-1,1]andy(x) \in C^1[-1,1]$.

As previously done in[32] the methodology adopted for the construction and the establishment of the noetherity of the operator L is based through the solvability of the first order linear differential equation defined by formula (1). This step will lead us to the analytical expression of the solutions of the considered equation in the needed functional space. We underline also that, as in the case $p = 1$, we bring out the central moment of the work underlying it connexion with the construction of the continuity of the regularisators, constructed before in our previous researches when we considered the interval [0,1], to the whole closed interval[-1,1]. See paper [12,32]. Note that, references on such approaches investigated by scientist Yurko V.A and many others with full details, may be found in [30,31,32]. Our final investigations through various computations and analysis will lead us to calculate step by step and to determine the deficient numbers of the operator L denoted $\alpha(L)$and $\beta(L)$. Consequently we define also from the previous the index $\varkappa(L)$of the operator L,depending of the various cases studied in connexion with the parameters $p \geq 2$ and also the sign and values of γ.

We organize the work as follow: first of all, we present in section 2the preliminaries related to the concept and the notions of well known noether theory. Section 3 is properly devoted to the presentation of the main results of this work and clearly is connected to the cases studied, with dependance of the sign and values of γ, in other words (related to cases a)$\gamma > 0$; b) $\gamma < 0$, c)$\gamma = 0$). And lastly, after an important remark we summarize the content of the work in section 4 titled conclusion, followed by some recommendations for the follow-up or future scientific works to undertake, stated in section 5.

2. Preliminaries

Before presenting in details our main results, the following definitions and concepts well known from the noether theory of operators are required. For details in full see also [6,10,12, 21,22].

By the way, we briefly review this important notions of Taylor derivatives, which is widely used when constructing noether theory of the considered operator A.

Let $C^m[-1,1], m \in \mathbb{Z}_+$, noted the Banach space of continuous functions on $[-1,1]$, having continuous derivatives up to order m, for which the norm is defined as following :

$$\|\varphi(x)\|_{C^m[-1,1]} = \sum_{j=0}^{m} \max_{-1\leq x\leq 1} \left|\varphi^{(j)}(x)\right| \qquad (2)$$

So, that we can consider $\varphi^{\{k\}}(0)$ are defined for all $k = 1,2,\dots,p$.

We define $C_0^{\{p\}}[-1,1]$ as a subspace of continuous functions, having finite Taylor derivatives up to order $p \in \mathbb{Z}_+$; and when $p = 0$, we put

$$(C_0^{\{p\}}[-1,1] = C[-1,1])). \qquad (3)$$

Let us also define a linear operator N^k on the space $C_0^{\{p\}}[-1,1]$ by the formula:

$$(N^k\varphi)(x) = \frac{\varphi(x) - \sum_{j=0}^{k-1} \frac{\varphi^{\{j\}}(0)}{j!} x^j}{x^k}, k = 1,2,\dots,p, (4)$$

One can easily verify the property $N^k = N^{k_1} N^{k-k_1}, 0 \leq k_1 \leq k, k, k_1 \in \mathbb{Z}_+$, where we put $N^0 = I$. The operator N^p characterizes the space $C_0^{\{p\}}[-1,1]$ as it can be seen from the following lemma see also [23, 28, 29].

Lemma 2.1. A function $\varphi(x)$belongs to$C_0^{\{p\}}[-1,1]$ if and only if, the following representation

$$\varphi(x) = x^p \phi(x) + \sum_{k=0}^{p-1} \alpha_k\, x^k, \qquad (5)$$

holds with $\phi(x) \in C[-1,1]$, and α_k being constants.

To prove Lemma 2.1 it is enough to observe that (5) implies that the Taylor derivatives of $\varphi(x)$ up to the order p exists, and $\varphi^{\{k\}}(0) = k!\,\alpha_k, k = 0,1,2,\dots,p-1, \varphi^{\{0\}}(0) = p!\,\phi(0)$ with $\phi(x) = (N^k\varphi)(x)$. Conversly, if $\varphi(x)$ belongs to $C_0^{\{p\}}[-1,1]$, and we define $\phi(x) = (N^k\varphi)(x)$ with $\alpha_k = \frac{\varphi^{\{k\}}(0)}{k!}, k = 0,1,2,\dots,p-1$,then the representation (5) holds. From Lemma 2.1, it follows that for $\varphi(x) \in C_0^{\{p\}}[-1,1]$ the inequality

$$\varphi(x) = x^p (N^k\varphi)(x) + \sum_{k=0}^{p-1} \frac{\varphi^{\{k\}}(0)}{k!} x^k, \qquad (6)$$

is valid.

Consequently, the linear operator N^p establishes a relation between the spaces $C_0^{\{p\}}[-1,1]$ and $C[-1,1]$. The space $C_0^{\{p\}}[-1,1]$ with the norm

$$\|\varphi\|_{C_0^{\{p\}}[-1,1]} = \|N^p\varphi\|_{C[-1,1]} + \sum_{k=0}^{p-1}|\varphi^{\{k\}}(0)|, \quad (7)$$

becomes a Banach space one.

Note that, it is obvious to see that $\|\varphi\|_{C[-1,1]} \leq$ φ *C0p−1,1.*

Finally, note that from the definition 2.1 it follows the following fact, if $\varphi(x)\epsilon C[-1,1]$, then $x^p\varphi(x) \in C_0^{\{p\}}[-1,1]$. This assertion may be generalized as follows.

Lemma 2.2. Let $p \in \mathbb{N}, s \in \mathbb{Z}_+$. If $\varphi(x)\epsilon C_0^{\{s\}}[-1,1]$ then, $x^p\varphi(x) \in C_0^{\{p+s\}}[-1,1]$, and the formula holds

$$\left(x^p\varphi(x)\right)^{\{j\}}(0) = \begin{cases} 0, \ j = 0,1,\dots,p-1, \\ \frac{j!}{(j-p)!}\varphi^{\{j-p\}}(0), j = p,\dots,p+s. \end{cases} \quad (8)$$

Proof. Note that a stronger assumption on the function

$\varphi(x)$, such that $\varphi(x) \in C_0^{\{p+s\}}[-1,1]$ would allow us to easily prove the lemma just by applying Leibniz formula.

For $s = 0$ the statement has already been proved above, so $x^p\varphi(x) \in C_0^{\{p\}}[-1,1]$, and $\left(x^p\varphi(x)\right)^{\{j\}}(0) = 0, j = 0,\dots,p-1$ and $\left(x^p\varphi(x)\right)^{\{p\}}(0) = p!\,\varphi(0)$. Now let us prove that $\left(x^p\varphi(x)\right)^{\{j\}}(0) = \frac{j!}{(j-p)!}\varphi^{\{j-p\}}(0), j = p+1,\dots\dots,p+s$. Since the derivatives are defined reccurently, and (8) is true for $j = p$, then it is sufficient to verify the passage from j to $j+1$. We have:

$$\left(x^p\varphi(x)\right)^{\{j+1\}}(0) = (j+1)!\lim_{x\to 0}\frac{x^p\varphi(x) - \sum_{l=p}^{j}\frac{x^l}{(l-p)!}\varphi^{\{l-p\}}(0)}{x^{j+1}} \quad (9)$$

$$= (j+1)!\lim_{x\to 0}\frac{\varphi(x) - \sum_{l=0}^{j-p}\frac{x^l\varphi^{\{l\}}(0)}{l!}}{x^{j+1-p}} = \frac{(j+1)!}{(j+1-p)!}\varphi^{\{j+1-p\}}(0). \quad (10)$$

Lemmas 2.1 and 2.2 imply the next lemma.

Lemma 2.3. Let $f(x) \in C_0^{\{p\}}[-1,1], p \in \mathbb{N}$ *and* $f(0) = \dots\dots =$ *fr−10=0, 1≤r≤p.* Then *f(x)xr∈C0p−s−1,1.*

It is also convenient to use an equivalent definition for the norm in $C_0^{\{p\}}[-1,1]$:

$$\|\varphi\|^1_{C_0^{\{p\}}[-1,1]} = \sum_{j=0}^{p}\|N^j\varphi\|_{C[-1,1]} \quad (11)$$

It is easy to verify the equivalence of the norms (7) and (11). Namely, we always have $|\varphi^{\{j\}}(0)| \leq j!\,\|N^j\varphi\|_{C[-1,1]}$ which gives the estimate $\|\varphi\| \leq c\|\varphi\|^1$. To obtain the inverse estimate, we use the next following equality below.

$$(N^j\varphi)(t) = t^{p-j}(N\varphi)(t) + \sum_{s=j}^{p-1}\frac{t^{s-j}}{s!}\varphi^{\{s\}}(0), \quad (12)$$

from which $\|N^j\varphi\|_{C[-1,1]} \leq \|\varphi\|, j = 0,1,\dots\dots,p-1$ and, then $\|\varphi\|^1 \leq c\|\varphi\|$.

Lemma 2.4. The operator $N^p: C_0^{\{p\}}[-1,1] \to C[-1,1]$ has the following properties:

1. N^p is bounded, and $\|N^p\varphi\|_{C[-1,1]} \leq \|\varphi\|_{C_0^{\{p\}}[-1,1]}$;

2. N^p is right invertible;

3. $\alpha(N^p) = p$, where $\alpha(N^p)$ is the dimension of the null subspace for N^p.

Proof. Statement 1) follows from the definition of the norm in (7). The invertibility is justified by the fact that the equation $N^p\varphi = f$ with an arbitrary $f(x) \in C[-1,1]$ has a solution $\varphi(x) = x^pf(x) \in C_0^{\{p\}}[-1,1]$, which follows from Lemma 2.1 and the equality $N^p(x^pf(x)) \equiv f(x)$. By 2) and noticing that $N^px^k = 0$ for all $k = 0,1,\dots.,p-1$, we arrive at to the point 3).

The lemma is proved.

Note that from the proof of this lemma, it follows that the equation $N^pf = g$ is always solvable in the space $C_0^{\{p\}}[-1,1]$ for every $g(x) \in C[-1,1]$, and its general solution has the form

$$f(x) = \sum_{k=0}^{p-1}c_k\,x^k + x^pg(x), \quad (13)$$

where c_k are arbitrary constants. Note that for the Taylor derivatives, many formulas are valid similar to those for ordinary derivatives; in particular, Leibniz formula, l'Hospitale rule and others see [13]. Nevertheless, for example, the formula

$$\left|\varphi^{(k)}(x)\right|^{\{r\}}(0) = \varphi^{\{k+r\}}(0) \quad (14)$$

is not always valid.

Next, let us move to the presentation of the general important results of the work in the following section.

3. Main Results

In this section, we undertake properly the noetherity investigation of the operator L.

Namely, here we consider as model to be investigated, the differential equation of the following type:

$$Ly(\mathrm{x}) = x^p y'(x) + \gamma y(x) = f(x)\ ;\ x \in [-1,1], \quad (15)$$

where

$p \in \mathbb{N}, p \geq 2\gamma \in$
$\mathbb{R}$and for the moment at the beginning $y(x) \in C^1[-1,1]$ with $f(x) \in C_0^{\{p\}}[-1,1]$.

As already mentionned the case $p = 1$ is completely investigated with full details in [32].

First of all, we consider the equation (1) in the case when $p = 2$.

$$Ly(\mathrm{x}) = x^2 y'(x) + \gamma y(x) = f(x)\ ;\ x \in [-1,1]. \quad (16)$$

We recall that when $x > 0$ the solution of the equation (16) has been wrote with dependance of the sign of γ see Section 2 in [23]. As in that case we use the classes of the functions $C_0^{s,\{p\}}[-1,1], p \geq s+1$.

A. The case $\gamma > 0$.

With respect to the results obtained in Section 2 from [23] and analysing the solutions of the homogeneous equation (16) we can formaly find out the solution in the form defined by:

$$y(x) = ce^{\frac{\gamma}{x}}_{-} + \begin{cases} e^{\frac{\gamma}{x}} \int_0^x e^{\frac{-\gamma}{t}} f(t) \frac{dt}{t^2} dt, x > 0 \\ \frac{f(0)}{\gamma}, \qquad x = 0 \\ e^{\frac{\gamma}{x}} \int_{-1}^x e^{\frac{-\gamma}{t}} f(t) \frac{dt}{t^2} dt, x < 0, \end{cases} \quad (17)$$

, where the function $e^{\frac{\gamma}{x}}_{-}$ is defined by the formula

$$e^{\frac{\gamma}{x}}_{-} = \begin{cases} 0, & x \geq 0, \\ e^{\frac{\gamma}{x}}, & x < 0. \end{cases} \quad (18)$$

Before we start analysing under which conditions onto the function $f(x)$ the solution of the equation (16) defined by the formula (18) will be existing in the class $C^1[-1,1]$, we note that under substitution (17) into (16) it takes place the equality.

Lemma 3.1. If the solution of the equation (16) in the space $C_0^{1,\{2\}}[-1,1]$ exists, then it is necessary that $f(x) \in C_0^{\{2\}}[-1,1]$. Under this condition it is supplementary needed that $y(x) \in C_0^{1,\{2\}}[-1,1]$ for every $\gamma \neq 0$.

Proof. We conduct the proof of this lemma basing onto the similar assertion from the section 2 see [23]. Infact, from the initial equation (16) we immediatly obtain $\gamma y(0) = f(0)$, from where and from the equation it follows that $\gamma y^{\{1\}}(0) = f^{\{1\}}(0)$. So that it is necessary that it exists $f^{\{1\}}(0)$, and more

$$y(0) = \frac{f(0)}{\gamma}, \qquad y^{\{1\}}(0) = \frac{f^{\{1\}}(0)}{\gamma} \quad (19)$$

with respect to (16) we have the following relationship

$$y'(x) + \gamma(N^2 y)(x) = (N^2 f)(x) \quad (20)$$

from where, passing to the limit when $x \to 0$, and taking into account that $y(x) \in C_0^{1,\{2\}}[-1,1]$ we obtain that $f(x) \in C_0^{\{2\}}[-1,1]$. From (20) it is also easy to see that if $f(x) \in C_0^{\{2\}}[-1,1], y(x) \in C^1[-1,1]$, then it is necessary that $y(x) \in C_0^{1,\{2\}}[-1,1]$. The lemma is proved.

Further, we note that it is not difficult to convince ourself that the function $y(x)$ in the form (17) is a solution of the equation (16) in the class $C_0^{1,\{2\}}[-1,1]$. We note that for $x \geq 0$ such proof has been realized before, so that it only remains to convince ourselves that and for $x < 0$, the obtained expression for the function $y(x)$ is a solution in the space $C_0^{1,\{2\}}[-1,1]$ and to verify, that $y'(x)$ is a continue function and it exists the taylor derivative $y^{\{2\}}(0)$.

Infact, with respect to the form of $y(x)$ we have under $x < 0$:

$$y(x) = \frac{f(0)}{\gamma} - \frac{f(0)}{\gamma} e^{\frac{\gamma}{x}} e^{\gamma} + e^{\frac{\gamma}{x}} \int_{-1}^x e^{\frac{-\gamma}{t}} [f(t) - f(0)] \frac{dt}{t^2} dt = A_1 + A2, \quad (21)$$

Where $A_1 = f(0)(1 - e^{\frac{\gamma}{x}+\gamma}) \to f(0)$ with respect to $\gamma > 0, x < 0$. As concerning the expression A_2, then we choose an arbitrary $\varepsilon > 0$ from the condition that $|f(t) - f(0)| < \varepsilon$ under $t > x_0$ and fixing the $x_0 < 0$. Then for $x_0 < x < 0$,

We obtain

$$|A_2| = \left| e^{\frac{\gamma}{x}} \int_{x_0}^x e^{\frac{-\gamma}{t}} [f(t) - f(0)] \frac{dt}{t^2} dt + e^{\frac{\gamma}{x}} \int_{-1}^{x_0} e^{\frac{-\gamma}{t}} [f(t) - f(0)] \frac{dt}{t^2} dt \right| \leq$$

$$\frac{\varepsilon}{|\gamma|} \left| 1 - e^{\frac{\gamma}{x} x_0 - \frac{\gamma}{x_0}} \right| + e^{\frac{\gamma}{x}} 2 \max_{-1 \leq t \leq x_0} |f(t)| \left| \frac{e^{\frac{\gamma}{x_0}} - e^{\gamma}}{\gamma} \right| \quad (22)$$

and we obtain, that $A_2 \to 0$ under $x \to -0$ with respect to arbitrary of ε. Therefore, taking into account (21) we reach to the fact that $\lim_{x\to -0} y(x) = \gamma^{-1} f(0)$. This, with respect to $y(0) = \gamma^{-1} f(0) = \lim_{x\to +0} y(x)$, we have proved the first relationship in (19).

As concerning the right relationship in the formula (19), then under $x < 0$ we have the following presentation:

$$\frac{y(x)-y(0)}{x} = \frac{ce^{\frac{\gamma}{x}}}{x} + \frac{1}{x}\left[e^{\frac{\gamma}{x}}\int_{-1}^{x} e^{\frac{-\gamma}{t}}[f(t)-f(0)]\frac{dt}{t^2}dt - \frac{f(0)}{\gamma}e^{\frac{\gamma}{x}}e^{\gamma}\right] =$$

$$\frac{ce^{\frac{\gamma}{x}}}{x} + e^{\frac{\gamma}{x}}\int_{-1}^{x} e^{\frac{-\gamma}{t}}\left[\frac{f(t)-f(0)}{t} - f^{\{1\}}(0)\right]\frac{dt}{xt} - \frac{f(0)}{x\gamma}e^{\frac{\gamma}{x}}e^{\gamma} +$$

f1(0)eγx−1xe−γtdtxt=A1+A2+A3+A4,

(correspondtly) (23).

For the investigation of the behavior of these terms under $x \to -0$, let us start with A_2. We have:

$$|A_2| \le \varepsilon e^{\frac{\gamma}{x}}\int_{x_0}^{x} e^{\frac{-\gamma}{t}}\frac{dt}{xt} + \left(\|Nf\|_{C[-1,0]} + |f^{\{1\}}(0)|\right)e^{\frac{\gamma}{x}}\int_{-1}^{x_0} e^{\frac{-\gamma}{t}}\frac{dt}{xt} \quad (24).$$

where $\varepsilon > 0$ has been chosen from the definition of $f^{\{1\}}(0)$ by the condition $|(Nf)(t) - f^{\{1\}}(0)| < \varepsilon$ under $x_0 < t < x$. It is not difficult to see that under $x \to -0$: $\lim_{x\to -0} e^{\frac{\gamma}{x}}\int_{x_0}^{x} e^{\frac{-\gamma}{t}}\frac{dt}{xt} = \frac{1}{\gamma}$ and $\lim_{x\to -0} e^{\frac{\gamma}{x}}\int_{-1}^{x_0} e^{\frac{-\gamma}{t}}\frac{dt}{xt} = 0$(25)

under fixed x_0. Consequently, $A_2 \to 0$ under $x \to -0$ with respect to the arbitrary ε.

Similarly to (25) we have $\lim_{x\to -0} A_4 = \frac{f^{\{1\}}(0)}{\gamma}$. With respect to $\lim_{x\to -0} A_1 = \lim_{x\to -0} A_3 = 0$, we conclude that $\lim_{x\to -0}\frac{y(x)-y(0)}{x} = y^{\{1\}}(0) = \frac{f^{\{1\}}(0)}{\gamma} = y'(0)$, and by the latter we achieve the verification of the truth of the relationship (19). Therefore the function $y(x)$ defined by the formula (17) is solution of the equation (16) in the space $C^1[-1,1]$. Let show that for the solution $y(x)$ it exists also $y^{\{2\}}(0)$. For this reason, let consider the presentation:

$$\frac{y(x)-y(0)-xy'(0)}{x^2} = \frac{1}{x^2}e^{\frac{\gamma}{x}}\int_{-1}^{x} e^{\frac{-\gamma}{t}}\frac{1}{x^2}\left[\frac{f(t)-f(0)-tf^{\{1\}}(0)}{t^2} - f^{\{2\}}(0)\right]dt + f^{\{2\}}(0)\frac{1}{x^2}e^{\frac{\gamma}{x}}\int_{-1}^{x} e^{\frac{-\gamma}{t}}dt + \frac{f(0)}{x^2}\left[e^{\frac{\gamma}{x}}\int_{-1}^{x} e^{\frac{-\gamma}{t}}\frac{dt}{t^2} - \frac{1}{\gamma}\right] + \frac{f^{\{1\}}(0)}{x^2}\left[e^{\frac{\gamma}{x}}\int_{-1}^{x} e^{\frac{-\gamma}{t}}\frac{dt}{t} - \frac{x}{\gamma}\right] + \frac{ce^{\frac{\gamma}{x}}}{x^2} = C_1 + C_2 + C_3 + C_4 + C_5, \quad (26)$$

where the first term C_1 can be approximated as previously, similarly to the approximation obtained for A_2.

Infact we have:

$$|C_1| \le \frac{\varepsilon}{x^2}e^{\frac{\gamma}{x}}\int_{x_0}^{x} e^{\frac{-\gamma}{t}}dt + \left(\|N^2 f\|_{C[-1,0]} + |f^{\{2\}}(0)|\right)\frac{1}{x^2}e^{\frac{\gamma}{x}}\int_{-1}^{x_0} e^{\frac{-\gamma}{t}}dt, \quad (27)$$

where $\varepsilon > 0$ has been chosen from the definition of $f^{\{2\}}(0)$ by the condition $|(N^2 f)(t) - f^{\{2\}}(0)| < \varepsilon$ under $x_0 < t < x$. Further, we have

$$\lim_{x\to -0}\frac{e^{\frac{\gamma}{x}}}{x^2}\int_{x_0}^{x} e^{\frac{-\gamma}{t}}dt = \frac{1}{\gamma}, \qquad \lim_{x\to -0}\frac{e^{\frac{\gamma}{x}}}{x^2}\int_{-1}^{x_0} e^{\frac{-\gamma}{t}}dt = 0 \quad (28)$$

under fixed x_0. Consequently, due to arbitrariness of $\varepsilon > 0$ and with respect to (27) and (28), we reach to the fact that $\lim_{x\to -0} C_1 = 0$. Similarly to the first term in (28) under fixed $x_0 = -1$ we obtain $\lim_{x\to -0} C_2 = \gamma^{-1} f^{\{2\}}(0)$.

For the consideration and analysis of the expression C_3, let denote through $I = e^{\frac{\gamma}{x}}\int_{-1}^{x} e^{\frac{-\gamma}{t}}\frac{dt}{t^2} - \frac{1}{\gamma} = -\frac{e^{\frac{\gamma}{x}}e^{\gamma}}{\gamma}$. Remarking that $C_3 = \frac{1}{x^2} I f(0)$, we have $\lim_{x\to -0} C_3 = 0$. Lastly, let compute limit C_4 under $x \to -0$, setting $C_4 = f^{\{1\}}(0) I'$, where $I' = \frac{1}{x^2}\left[e^{\frac{\gamma}{x}}\int_{-1}^{x} e^{\frac{-\gamma}{t}}\frac{dt}{t} - \frac{x}{\gamma}\right]$. Integrating by parts and considering the first term in the relationship (28) , we find out $\lim_{x\to -0}\gamma^2 I' = -1$. With respect to these computations we reach to the equality $\lim_{x\to -0} C_4 = -\gamma^{-2} f^{\{1\}}(0)$. It is clear that $\lim_{x\to -0} C_5 = 0$. Unifying all the previous and passing to the limit in the left hand and right sides (28), we obtain definitively that $y^{\{2\}}(0)$ exists and

$$\lim_{x\to -0}\frac{y(x)-y(0)-xy'(0)}{x^2} = y^{\{2\}}(0) = \frac{f^{\{2\}}(0)}{\gamma} - \frac{f^{\{1\}}(0)}{\gamma^2}. \quad (29)$$

If we turn to the equation (16), then we can rewrite it in the following way $y'(x) = -\frac{\gamma}{x^2}y(x) + \frac{f(x)}{x^2}$, from where, considering (19) and passing to the limit when $x \to -0$ in (20), we obtain $\lim_{x\to -0} y'(x) + \gamma y^{\{2\}}(0) = f^{\{2\}}(0)$. From the previous, it follows $\lim_{x\to -0} y'(x) = f^{\{2\}}(0) - \gamma y^{\{2\}}(0) = \frac{1}{\gamma} f^{\{1\}}(0) = y'(0)$ i.e $y'(x)$ is a continuous function on the interval $[-1,1]$ and $\lim_{x\to -0} y'(x) = y'(0) = y^{\{1\}}(0)$.

Theorema 3.1. Let $\gamma > 0$ and $f(x) \in C_0^{\{2\}}[-1,1]$. Then the equation (16) is solvable in the space $C_0^{1,\{2\}}[-1,1]$ and the solution is not unique and defined by the formula (17), where c is an arbitrary constant.

Analogously, we can investigate a more general equation (15) under $p > 1$.

Theorema 3.2. Let $\gamma > 0$ and $f(x) \in C_0^{\{p\}}[-1,1], p \geq 2$. Then, the equation (15) is solvable in the space $C_0^{1,\{p\}}[-1,1]$ and having a solution defined by the following formula:

$$y(x) = c\, e^{\frac{\gamma}{p-1}x^{1-p}}{}_{-} + \begin{cases} e^{\frac{\gamma}{p-1}x^{1-p}} \int_0^x e^{-\frac{\gamma}{p-1}t^{1-p}} f(t)\frac{dt}{t^p}, x > 0, \\ \frac{f(0)}{\gamma}, \quad x = 0, \\ e^{\frac{\gamma}{p-1}x^{1-p}} \int_{-1}^x e^{-\frac{\gamma}{p-1}t^{1-p}} f(t)\frac{dt}{t^p}, x < 0, \end{cases} \quad (30)$$

where c is an arbitrary constant.

The operator L defined by the left hand side (15) is a noether operator with the index $\varkappa(L) = 1$ and $d-$characteristic$(1,0)$.

Case $\gamma = 0$. In this namely case it is obvious true the following assertion.

Theorem 3.3. Let $\gamma = 0$ and $f(x) \in C_0^{\{p\}}[-1,1]$ and more, $f(0) = f^{\{1\}}(0) = \ldots.. = f^{\{p-1\}}(0) = 0$. Then the equation (15) under $\gamma = 0$ in the space $C^1[-1,1]$ is solvable and the solution is defined by the following formula:

$$y(x) = \int_{-1}^x (N^p)(t)dt + c. \quad (31)$$

For the operator L defined by the left hand side of the equation (15) under $\gamma = 0$, we obtain that it is a noether operator with the index $\varkappa(L) = 1 - p$ and $d-$characteristic$(1, p)$.

Case $\gamma < 0$. From a formal type of the solution of the equation (16) it is clear that by the virtue of the divergence of the integrals in the formula (17) it is necessary to take the particular solution in another way and, the general solution of the equation (16) is defined by:

$$y(x) = ce^{\frac{\gamma}{x}}{}_{+} + \begin{cases} -e^{\frac{\gamma}{x}} \int_x^1 e^{\frac{-\gamma}{t}} f(t)\frac{dt}{t^2}, x > 0, \\ \frac{f(0)}{\gamma}, \quad x = 0, \\ -e^{\frac{\gamma}{x}} \int_x^0 e^{\frac{-\gamma}{t}} f(t)\frac{dt}{t^2}, x < 0, \end{cases} \quad (32)$$

where the function $e^{\frac{\gamma}{x}}{}_{+}$ is defined by the formula :

$$e^{\frac{\gamma}{x}}{}_{+} = \begin{cases} e^{\frac{\gamma}{x}}, & x \geq 0, \\ 0, & x < 0. \end{cases} \quad (33)$$

It is not difficult to verify that the function $y(x)$ defined by the formula (32) is a solution of equation (16) on the interval $[-1,1]$. Analogously as in the case $\gamma > 0$, we can prove the thruth of the analogue of lemma 3.1. Not stating all details of such proof let us formulate the following theorem.

Theorema 3.4. Let $\gamma < 0$ and $f(x) \in C_0^{\{2\}}[-1,1]$. Then the equation (15) is solvable in the space $C_0^{1,\{2\}}[-1,1]$ and the solution is not unique defined by the formula (32), where c is an arbitrary constant.

In conclusion to this section, note that for the equation of the general form under $p \geq 2$, we can obtain analogue theorem.

Theorema 3.5. Let $\gamma < 0$ and $f(x) \in C_0^{\{p\}}[-1,1]$. Then the equation (15) is solvable in the space $C_0^{1,\{p\}}[-1,1]$ under every right hand side and the solution is defined by the following formula:

$$y(x) = c\, e^{\frac{\gamma}{p-1}x^{1-p}}{}_{+} + \begin{cases} -e^{\frac{\gamma}{p-1}x^{1-p}} \int_x^1 e^{-\frac{\gamma}{p-1}t^{1-p}} f(t)\frac{dt}{t^p}, \quad x > 0, \\ \frac{f(0)}{\gamma}, \quad x = 0, \\ -e^{\frac{\gamma}{p-1}x^{1-p}} \int_x^0 e^{-\frac{\gamma}{p-1}t^{1-p}} f(t)\frac{dt}{t^p}, \quad x < 0, \end{cases} \quad (34)$$

where c is an arbitrary constant.

Next let us pass to the question related to the investigation of the noetherity of the operator $L: C_0^{1,\{2\}}[-1,1] \to C_0^{\{2\}}[-1,1]$, defined by the left right hand side of the equation (16). Besides with the operator L let introduce the operator $L^{-1}: C_0^{\{2\}}[-1,1] \to C_0^{1,\{2\}}[-1,1]$, which put into correspondance the function $f(x) \in C_0^{\{2\}}[-1,1]$ with dependance to the sign of γ the following expressions:

a) $\gamma > 0$, $(L^{-1}f)(x) = \begin{cases} e^{\frac{\gamma}{x}} \int_0^x e^{\frac{-\gamma}{t}} f(t) \frac{dt}{t^2}, & x > 0, \\ \frac{f(0)}{\gamma}, & x = 0, \\ e^{\frac{\gamma}{x}} \int_{-1}^x e^{\frac{-\gamma}{t}} f(t) \frac{dt}{t^2}, & x < 0, \end{cases}$ (35)

b) $\gamma < 0$, $(L^{-1}f)(x) = \begin{cases} -e^{\frac{\gamma}{x}} \int_x^1 e^{\frac{-\gamma}{t}} f(t) \frac{dt}{t^2}, & x > 0, \\ \frac{f(0)}{\gamma}, & x = 0, \\ -e^{\frac{\gamma}{x}} \int_x^0 e^{\frac{-\gamma}{t}} f(t) \frac{dt}{t^2}, & x < 0. \end{cases}$ (36)

C) The noetherity of the operator L in the case $\gamma > 0$.

As it has been done in the section 2 [23] in the case of the interval [0,1] also see [23]we can bring out the following lemmas:

Lemma 3.2. The operator $L: C_0^{1,\{2\}}[-1,1] \to C_0^{\{2\}}[-1,1]$ is bounded.

Lemma 3.3. The operator $L^{-1}: C_0^{\{2\}}[-1,1] \to C_0^{1,\{2\}}[-1,1]$ is bounded.

Proof. By definition, the boundeness of the operator L^{-1} in the space $C_0^{1,\{2\}}[-1,1]$ conduct us to the computation of the following norm

$$\|L^{-1}f\|_{C_0^{1,\{2\}}[-1,1]} = \|L^{-1}f\|_{C^1[-1,1]} + \|N^2L^{-1}f\|_{C[-1,1]}. \quad (37).$$

The approximation of each term is conducted similarly as it has been done in the case of the interval [0,1], but more difficultly. Such details related to this matter can been found in [43].

Theorem 3.6. The operator $L: C_0^{1,\{2\}}[-1,1] \to C_0^{\{2\}}[-1,1]$ is invertible in the subspace distinguished by the condition$y(-1) = 0$.

c) **Proof.** The factthat $LL^{-1}f = f$, factly is verified before with respect that $y(x) = (L^{-1}f)(x)$ is a solution. Next calculate under $x < 0$

$$L^{-1}Ly = e^{\frac{\gamma}{x}} \int_{-1}^x e^{\frac{-\gamma}{t}} y'(t)dt + \gamma e^{\frac{\gamma}{x}} \int_{-1}^x e^{\frac{-\gamma}{t}} y(t) \frac{dt}{t^2}, \quad (38).$$

Integrating part by part in the first integral, we have $L^{-1}Ly = y(x) - y(-1)e^{\frac{\gamma}{x}}e^{\gamma} = (I - T_1)y$, where $(T_1y)(x) = y(-1)e^{\frac{\gamma}{x}}e^{\gamma}$ is a projection (under $x < 0$). Infact, $(T_1^2y)(x) = T_1(y(-1)e^{\frac{\gamma}{x}}e^{\gamma}) = y(-1)e^{\frac{\gamma}{x}}e^{\gamma} = (T_1y)(x)$.

So that, for the operator L andL^{-1} with respect to the previous ideas, it takes place the following representations:

$$L^{-1}Ly = I - T_1y, \qquad LL^{-1}f = f, f(x) \in C_0^{\{2\}}[-1,1], \quad y \in C_0^{1,\{2\}}[-1,1], \quad (39)$$

where

$(T_1y)(x) = y(-1)e^{\gamma}e^{\frac{\gamma}{x}}$_ is one a dimensional projection.

d) **Theorem 3.7.** Let $\gamma > 0$. The operator $L: C_0^{1,\{2\}}[-1,1] \to C_0^{\{2\}}[-1,1]$ is noether with the index $\varkappa(L) = +1$ and deficient numbers (1,0).

The proof of this theorem is deduced from the previous conducted ideas and theorem 3.6.

Case $\gamma < 0$. Analogously, we can investigate the question of the noetherity of the operator L in the case when $\gamma < 0$. We will just give the important moments in this case. From our previous results ralated to the boundedness of the operator L in the case when $\gamma > 0$, we can remark that in the situation when $\gamma < 0$, the operator L is bounded.

The fact that the operator L^{-1} defined by the formula (36) is right invertible is beyong doubt as the equality

$$LL^{-1}f = f, \quad f(x) \in C_0^{\{2\}}[-1,1](40).$$

obtained in the process of the verification of the solution. Analogously as in the case when $\gamma > 0$, we can bring out the following results:

Lemma 3.4. The operatorL^{-1} of the form (36) is bounded from $C_0^{\{2\}}[-1,1]$ in $C_0^{1,\{2\}}[-1,1]$.

Theorem 3.8. Let $\gamma < 0$. The operator $L: C_0^{1,\{2\}}[-1,1] \to C_0^{\{2\}}[-1,1]$ is invertible into the sub space distinguished by the condition $y(1) = 0$.

Proof. We have under $x < 0$:

$$L^{-1}Ly = -e^{\frac{\gamma}{x}} \int_x^0 e^{\frac{-\gamma}{t}} y'(t)dt - \gamma e^{\frac{\gamma}{x}} \int_x^0 e^{\frac{-\gamma}{t}} y(t) \frac{dt}{t^2}, \quad (41).$$

The integration part by part in the first term gives $(L^{-1}Ly)(x) = y(x), x < 0$. Concerning the value $x > 0$, similarly to (41) we have

$$L^{-1}Ly = -e^{\frac{\gamma}{x}}\int_x^1 e^{\frac{-\gamma}{t}}y'(t)dt - \gamma e^{\frac{\gamma}{x}}\int_x^1 e^{\frac{-\gamma}{t}}y(t)\frac{dt}{t^2}, \quad (42).$$

Analogously, integrating part by part in the first term, we obtain $(L^{-1}Ly)(x) = -y(1)e^{\frac{\gamma}{x}}e^{-\gamma} + y(x)$ under $x > 0$.

So that for the operators L and L^{-1} it is holding place the following representations (39) where at this time $(T_1y)(x) = y(1)e^{-\gamma}e^{\frac{\gamma}{x}}{}_+$ and T_1 is an onedimensional projection. That is ending the proof.

Theorem 3.9. Let $\gamma < 0$. The operator $L : C_0^{1,\{2\}}[-1,1] \to C_0^{\{2\}}[-1,1]$ is noether operator with the index $\varkappa(L) = +1$ and deficient numbers (1,0).

The proof of this theorem is followed from the ideas used in the process of the construction of the noetherity and from theorem 3.4.

Analogously, we can obtain the assertion in the general situation of the case when $p \geq 2$.

Theorema 3.10. The operator $L : C_0^{1,\{p\}}[-1,1] \to C_0^{\{p\}}[-1,1]$ under $\gamma \neq 0$ and the operator $L : C^1[-1,1] \to C_0^{\{p\}}[-1,1]$ under $\gamma = 0$, $p \geq 2$ defined by the formula (15) where $f(x) \in C_0^{\{p\}}[-1,1], \gamma \in \mathbb{R}$ is a noether operator, and more for the index and deficient numbers, it is holding place the relationships:

$$\begin{cases} a) \ \ \gamma \neq 0, & \varkappa(L) = 1, & (\alpha,\beta) = (1,0), \\ b) \ \ \gamma = 0, & \varkappa(L) = 1 - p, & (\alpha,\beta) = (1,p). \end{cases} (43).$$

4. Conclusion

This achieved scientifc work presents in full details the completed investigation of the establishment and the construction of noetherity theory for the operator L, defined by a first order linear singular differential equation depending of the parameters $p \in \mathbb{N}, p \geq 2$, $\gamma \in \mathbb{R}$ in the space of continuous functions $C^1[-1,1]$.

We have found the solvability condition of the equation (1) taking into account the nature of the parameters $p \in \mathbb{N}, p \geq 2, \gamma \in \mathbb{R}$ in various cases. The previous done work at this step lead us to determine the deficient numbers of the considered operator L denoted $(\alpha,\beta)$and, and from that therefore is deduced it index $\varkappa(L)$, which is finite in all cases, making clearly the operator L to be noether. The content of the results of the whole investigation

5. Recommendations

It is clear that one can note that, we have realized the most important investigation necessary to undertake the investigation for noetherity question of an integro-differential operator, defined by a third kind singular integral equation which, has the considered studied operator L as main part. From the general theory of noetherity construction for operators, we know that under perturbation of a noether operator by a compact operator and, in the case to be investigated in a brief future, we will reach and maintain the noetherity nature of the initial oparator L. This goal will be the next future work to undertake when we do consider first of all, the operator A as a sumof two operators L and K where, L is the operator defined by $Ly(\mathrm{x}) = x^p y'(x) + \gamma y(x) = f(x)$ and K is a compact operator defined as follows $K\varphi = \int_{-1}^{1} k(x,t)y(t)dt$.

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