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On Prime Numbers Generation and Pairing

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Abstract: *The present paper tries to develop a simple scheme to generate primes from the elements of the prime natural number set. The present procedure counts on a recursive algorithm to generate natural numbers by adding powers of 2 to previous subsets. Raw results indicate that starting from a given odd natural number, one can obtain a prime number sequence in this way. Then, one can consider a prime number as the result of summing a sum of powers of 2 to a previous odd number. Finally, one can conjecture that all prime numbers can result from previous prime numbers added to an element or some sum of the* $\mathbf{2}^{\mathbb{N}}$ *set elements.*

Keywords: *Mersenne Numbers; Pre-Mersenne Numbers; The set* $\mathbf{2}^{\mathbb{N}}$ *; Generation of Natural Numbers; Generation of Primes; Extended Twin Primes; Paired Primes; Prime Source; Conjecture on Primes; Prime Numbers Generation Algorithm;*

1. Introduction

No doubt that prime numbers [[1]] are the stars of natural number theory and beyond, considering the connection of the Riemann hypothesis [[1],[2]] with this subset of natural numbers. Still today, several recent papers are concerned with the structure of prime numbers. For instance, the references [[2],[4],[4],[6],[7],[8]] show an interest in prime numbers in the literature. Moreover, prime numbers are the backbone of the Collatz conjecture, as recently demonstrated [[9]].

1.1. Mersenne Numbers

On the other hand, the present study enhances the role of Mersenne numbers, which are a subset of odd natural numbers defined as:

$$\forall N \in \mathbb{N} : M(N) = 2^N - 1. \tag{1}$$

All Mersenne numbers bear an *N*-dimensional binary representation made of a bit string entirely made of unit bits, that is:

$$M(N) \leftrightarrow \langle \mathbf{1}_N | = (1,1,\ldots,1)_N \Rightarrow \sum_{I=1}^{N} 2^{I-1} = 1 + 2 + 2^2 + \ldots 2^{N-1} \tag{2}$$

Some of the Mersenne numbers are primes; see reference [[10]] for their commented list.

Previously it has been put into evidence Mersenne numbers' capital role from two points of view. First, they have been described as the milestones to recursively generate natural numbers, such as in the references [[11],[12]]. Second, determining the close relationship between Mersenne numbers and the number of primes contained in a Mersenne interval [[11]].

1.2. Mersenne Intervals

This paper also uses the concept of a Mersenne *interval* symbolized by: $\mathbf{M}(N)$. One can define it as a natural number subset of cardinality 2^N constructed as follows:

$$\mathbf{M}(N) = \{0,1,2,\ldots,M(N)\} \subset \mathbb{N}. \tag{3}$$

Including the number 0 in the sequence defining a Mersenne interval and, therefore, in the natural number set $\mathbb{N}$ is compulsory. Accepting the 0 element as a natural number is needed to allow a complete Mersenne interval transcript into the set of binary strings made by the set of vertices of an *N*-dimensional Boolean hypercube. See, for instance, references [[13],[14],[15]].

1.3. The set of Powers of 2 and Mersenne numbers

Also, the present paper will try to evidence the role of the set of powers of 2:

$$\mathbf{2}^{\mathbb{N}} = \{1,2,4,\ldots,2^N,\ldots\}, \tag{4}$$

and the sums of its elements to generate prime numbers starting from any odd or prime natural number.

The Mersenne primes sequence [[10]] corresponds to an example of prime numbers generation via collaboration on the internet, where all the elements known up to date are listed. The largest prime known is a Mersenne number, the 51st on the list: $2^{82589933} - 1$, a prime number still to be confirmed with 24862048 digits that cannot be printed so easily here (or anywhere).

However, all Mersenne numbers have an attractive property that one can easily express as:

$$\forall N \in \mathbb{N} : 2^N + M(N) = 2^N + (2^N - 1) = 2 \cdot 2^N - 1 = 2^{N+1} - 1 = M(N+1) \tag{5}$$

Indicating that two Mersenne numbers, independently if they are prime or not, can be connected by a sum of powers of 2. That is:

$$\forall\{M,N\}\subset\mathbb{N}\wedge M<N: \quad M(N)=M(M)+\sum_{I=0}^{N-M-1}2^{M+I} \qquad (6)$$

This property of Mersenne numbers has inspired the present study.

It is also interesting to note that the ordered exponents providing Mersenne primes produce three differentiated linear plots when their logarithms are plotted versus the corresponding order numbers. This empirical feature indicating the progression of known Mersenne primes presents diverse sectors as the exponents increase.

1.4. Organization of this Study

One will organize this work by describing the generation structure of prime numbers as follows. First, the generation of natural numbers will be described from the point of view of a natural vector space. Then the role of the set $\mathbf{2}^{\mathbb{N}}$ will be discussed as a generator of prime numbers. Next, an algorithm will be described to generate prime numbers from summing elements of the set $\mathbf{2}^{\mathbb{N}}$ to the odd natural number elements. Finally, a conjecture about the generation of prime numbers will be presented before the conclusions.

2. Generation of Natural Numbers and Vector Spaces

Suppose a natural vector semi space[1]: $\mathbf{V}_{2^N}(\mathbb{N})$. A row vector over it can be associated with a 2^N - dimensional well-ordered vector, whose elements correspond to a choice of natural numbers in a Mersenne interval:

$$\mathbf{M}(N)\subset\mathbb{N}\rightarrow \quad \mathbf{M}(N)=\{0,1,\ldots,2^N-2,2^N-1\}. \qquad (7)$$

Such a numerical sequence can be expressed as: $\langle\mathbf{M}_N|\in\mathbf{V}_{2^N}(\mathbb{N})$, where the bra Dirac's notation describes row vectors. The upper limit element of $\mathbf{M}(N)$ is the *N*-th Mersenne number.

The previous to the last element in a Mersenne interval $\mathbf{M}(N)$ is an even *pre-Mersenne number*:

$$P(N)=2^N-2=M(N)-1=2M(N-1). \qquad (8)$$

[1]Here the term *semi space* is used as a synonym of *orthant*.

The *N*-th power: $2^N=M(N)+1$ might be seen as the *N*-th *post-Mersenne number* and corresponds to the first number outside a given Mersenne interval, greater than all the $\mathbf{M}(N)$ interval elements.

One might construct such kind of vectors by a recursive procedure involving a direct sum, like:

$$\langle\mathbf{M}_{N+1}|=\langle\mathbf{M}_N|\oplus\left(2^N\langle\mathbf{1}_{2^N}|+\langle\mathbf{M}_N|\right), \rightarrow\langle\mathbf{M}_{N+1}|\in\mathbf{V}_{2^{N+1}}(\mathbb{N}) \qquad (9)$$

where one can write the second direct sum term vector as:

$$\langle\mathbf{A}_N|=2^N\langle\mathbf{1}_{2^N}|+\langle\mathbf{M}_N| \qquad (10)$$

and plays a relevant role in the Collatz conjecture solution [[9],[16],[17],[18]]. One can see that the first element of the vector $\langle\mathbf{A}_N|$ will be 2^N because the first element of the vector $\langle\mathbf{M}_N|$ representing $\mathbf{M}(N)$ is 0. This fact justifies the inclusion of 0 as the lower bound of any Mersenne interval and accepting 0 as an element of the natural number set.

The algorithm (9) also indicates that in every subsequent step of the recursion, a vector part corresponds to the previous step vector, while the newly generated elements are those defined by the vector $\langle\mathbf{A}_N|$ in the equation (10).

The recursive construction of vectors in the equation (9) encompasses the whole set of natural numbers when: $N\rightarrow\infty$. At every iterative step, the vector space dimension doubles, as goes from 2^N to 2^{N+1}, which means that the cardinality of the generated Mersenne intervals doubles at each recursive step.

Then, the details of the generated natural numbers as vectors appear in the same fashion as the previous description of the generation algorithm of natural numbers from the point of view of set theory [[11]].

Therefore, one can conclude that one can recursively generate the set of natural numbers with the definition of successive Mersenne intervals and the set of powers $\mathbf{2}^{\mathbb{N}}$. Thus, Mersenne numbers are essential in generating natural numbers in the company of the powers of 2.

3. The Set $\mathbf{2}^{\mathbb{N}}$ and Prime Numbers Generation

The vector $\langle\mathbf{A}_N|$ in the equation (10) indicates how one could understand the successive natural numbers generated via the recursive algorithm in the equation (9). To observe the generated numbers, first, one can divide the natural number set into the usual parts made by the even $\mathbb{E}$ and odd $\mathbb{O}$ numbers, that is: $\mathbb{N} = \mathbb{E} \bigcup \mathbb{O} \wedge \mathbb{E} \bigcap \mathbb{O} = \varnothing$. Then one assumes that the only even prime number 2 belongs to $\mathbb{E}$. The rest of the prime numbers $\mathbb{P}$ are odd. Thus, they are members of $\mathbb{O}$: $\mathbb{P} \subset \mathbb{O}$. The following will not consider this distinction between primes involving the even prime number 2. Thus, when speaking of the set of primes, one will only refer to the set $\mathbb{P}$ of odd primes.

After the previous description, using the recursive generation of natural numbers via the equation (9), the elements of the vector construction at each step shown in the equation (10)can be analyzed as follows:

a) Even numbers of vector $\langle \mathbf{M}_N |$ components transform into even numbers of vector $\langle \mathbf{A}_N |$ components.

b) Odd numbers of vector $\langle \mathbf{M}_N |$ components transform into odd numbers of vector $\langle \mathbf{A}_N |$ components.

Therefore, one can be sure that, in case that some odd components of the vector $\langle \mathbf{A}_N |$ are primes, one can express them in general as:

$$\begin{aligned} &\forall a_{NI} \in \langle \mathbf{A}_N | \wedge a_{NI} \in \mathbb{P} \Rightarrow \\ &\quad \exists m_{NI} \in \langle \mathbf{M}_N | \wedge m_{NI} \in \mathbb{O} : \\ &\quad a_{NI} = 2^N + m_{NI} \end{aligned} \tag{11}$$

so new primes not contained as components in the vector $\langle \mathbf{M}_N |$ are generated by the equation (10) algorithm via the already generated primes or from odd numbers already contained in $\langle \mathbf{M}_N |$. Somehow this has been debated in previous work [[12]].

In the present author's opinion, the subject of this study, that is, how prime numbers are generated, has not yet been analyzed in-depth, which is the main reason for the present discussion. One can structure the analysis of the sequential construction of prime numbers as a consequence of the previous step sequences concerning the Mersenne intervals generation through the algorithm (9).

The equation (11) also indicates that the newly generated prime numbers will essentially depend on the elements of the power set $\mathbf{2}^{\mathbb{N}}$. Such conclusion was the subject sketched in reference [[11]], and the main body of the discussion, contained in the reference [[12]], extended the concept of twin prime numbers.

4. Paired Primes and the Prime Number Generation Algorithm

4.1. Introduction to Prime Numbers Generation

After knowing that one can create Mersenne numbers with sums of powers of 2, as discussed at the beginning, one can start the analysis of the sequential prime number generation by observing first the simple sum equalities:

$$\begin{aligned} &3+2=5 \;:\; 3+2^2=7 \;: \\ &3+2^3=11 \;:\; 3+4^4=19 \end{aligned} \tag{12}$$

where in red are marked the generated prime numbers $\{5,7,11,19\}$ starting from 3. One can also use a different generation using sums of powers of 2, like in the sequence starting from 5, generating the previous primes except 5, which is converted now in the first term.

$$5+2+2^2+2^3=7+4+8=11+8=19 . \tag{13}$$

One can further explore such a sequence of powers of 2 sums, taking the prime 19 as a new first step:

$$\begin{aligned} &19+4+8+16+32+64+128+256+512= \\ &23+8+16+32+64+128+256+512= \\ &31+16+32+64+128+256+512= \\ &47+32+64+128+256+512= \\ &79+64+128+256= \\ &143+128+256+512= \\ &271+256+512= \\ &527+512= \\ &1039 \end{aligned} \tag{14}$$

where the generated numbers are red, marking the primes, and the blue color signals the odd composites. The generated prime set is now: $\{23,31,47,79,271,1039\}$. In previous work, these primes on this set were named *extended twin primes* [[12]], one of which is the Mersenne prime {31}. The gaps between extended twin primes correspond to a sequence of powers of 2. Alternatively, one can associate the generated pairs with sums of these powers.

4.2. Generation of Arbitrary Prime Numbers

Such generating sequences can also be obtained with arbitrary starting numbers, even if they are more significant than the previous examples in equations (12) and (14). One can start with an arbitrarily chosen composite number, for example:

$21512641057 = 317 \times 2113 \times 32117$,

and using the same color code as before, one can write two equalities:

$$21512641057 + 2^{18} = 21512903201$$
$$21512903201 + 2^{23} = 21521291809 \qquad (15)$$

Anyone can obtain examples of large composite and prime numbers yielding prime numbers after adding to these elements' powers of 2. One can quickly reach a web program that can be helpful for this purpose [[19]].

Starting from a composite number, one arrives, adding a high power of 2 or a sum of powers of 2 to different prime numbers. The same occurs with the prime number obtained with the previous composite number treatment. When dealing with numbers possessing many digits, it seems logical to consider that the apparition of prime numbers needs more extensive powers of 2 to generate new primes.

However, except for the need for significantly extensive powers of 2, nothing seems to oppose the apparition of new prime numbers when applying the equation (9) algorithm. After all, the number of primes has been demonstrated, see reference [[11]], increasing, as the Mersenne interval corresponds to a larger Mersenne number upper bound.

Furthermore, the addition of sums of powers of 2 can provide similar, but not the same, results, like in the examples below:

$$21512641057 + \sum_{I=1}^{12} 2^I = 21512649247$$
$$21512903201 + \sum_{I=1}^{12} 2^I = 21512911391 \qquad (16)$$

One can replace the sum of powers of two $S(12)$ by the pre-Mersenne number $P(13)$, as discussed below.

4.3. Some Vocabulary about Generated Primes

Thus, one can define some new terms associated with the generation of prime numbers via the algorithm of the equation (9). Suppose one has the following situation:

$$\exists x \in \mathbb{O} \wedge \exists 2^N \in \mathbf{2}^{\mathbb{N}} \Rightarrow p = 2^N + x \wedge p \in \mathbb{P} \qquad (17)$$

In this case, one can say that the odd number x is the *source* of the prime number p with a *gap* 2^N. The equation (17) can be more restricted, and one can use as a source q a prime number, too, that is:

$$\exists q \in \mathbb{P} \wedge \exists 2^N \in \mathbf{2}^{\mathbb{N}} \Rightarrow p = 2^N + q \wedge p \in \mathbb{P}. \qquad (18)$$

In this latter case, the generated prime p is an *extended twin prime* concerning the source q, which can now be named a *paired prime*. One can also say that the primes' pair: $\{p, q\}$ is *paired* with a gap of 2^N.

4.4. Sums of Powers of 2 and Pre-Mersenne Numbers

A note about the sums of powers of 2 might be helpful now. There is no need to compute these sums as one can condense them into a pre-Mersenne number of the appropriate order. The sums of ordered powers of 2, which belong to some sequence obtained from the set: $\mathbf{2}^{\mathbb{N}} = \{1, 2, 2^2, \dots, 2^N, \dots\}$, is the corresponding pre-Mersenne number, as one can write:

$$\sum_{I=1}^{N} 2^I = 2 + 4 + 8 + \dots 2^N =$$
$$S(N) = 2^{N+1} - 2 = P(N+1) = 2M(N) \qquad (19)$$

which generates the sequence:

$$\{2, 6, 14, 30, \dots (2^{I+1} - 2), \dots\},$$

corresponding to the ordering: $I = 1, 2, 3, \dots$

Accordingly, as briefly commented in the equation (16), one can substitute the two sums $S(12)$ with the pre-Mersenne number $P(13)$. The same pre-Mersenne number appears in both prime generation equalities in this example, but evidently, one cannot consider this a general case. As additional examples, one can present two applications of the pre-Mersenne numbers that one can write as:

$$21521291807 + P(17) = 21521422877, \qquad (20)$$

which shows how large prime numbers can be paired via sums of powers of 2.

Moreover, using now the odd composite number obtained arbitrarily:

$21521291817 = 3 \times 19963 \times 359353$,

as a source, one can also generate the following:

$$21521291817 + P(78) =$$
$$302231454903678814968359 \qquad (21)$$

which provides a large prime number. In this case, adding the sum of powers of 2 yields a prime number with an even more significant number of digits.

One can consider the prime set:

$$\{23, 31, 47, 79, 271, 1039\},$$

generated in the equation (14), as a paired set of primes with diverse gaps. However, one can consider these gaps with sums of powers of 2 and replace them with adequate pre-Mersenne numbers. In these cases, the gaps of the paired primes will be pre-Mersenne numbers:

$$P(N) = 2^N - 2 = S(N-1) = 2M(N-1)$$

instead of post-Mersenne numbers of the kind 2^N.

Thus, the generation of prime numbers using sums like in equations (14) and (16), using the pre-Mersenne numbers, corresponds to another kind of paired primes.

Moreover, one can express a sum of ordered powers of 2 starting at an arbitrary power as a difference of pre-Mersenne numbers, that is:

$$\forall M < N : S(M,N) = \sum_{I=M}^{N} 2^I = \left(\sum_{I=1}^{N} 2^I\right) - \left(\sum_{I=1}^{M-1} 2^I\right) = S(N) - S(M-1) \quad (22)$$

One can also write the corresponding power of two in this way:

$$\begin{aligned} 2^N &= M(N)+1 = S(N-1,N) = \\ &S(N)-S(N-1) = 2^{N+1}-2-2^N+2 = \\ &2^{N+1}-2^N = 2^N(2-1) \Rightarrow 2^N \end{aligned} \quad (23)$$

such a result clearly shows that the powers of 2 can also be called post-Mersenne numbers.

Therefore, one can associate the gaps existing in paired and pre-Mersenne primes to the same kind of natural numbers obtained with sums of the elements of the power set: $\mathbf{2}^{\mathbb{N}}$.

One has discussed that prime numbers can be found by summing powers of two or pre-Mersenne numbers to prime numbers.

4.5. Generalizing the Pre-Mersenne Numbers

Pre-Mersenne numbers are the first term of a sequence of even numbers, appearing as elements in the vector $\langle \mathbf{A}_N |$ defined in the equation (10), which one can write as:

$$\left\{2^N - 2^1;\ 2^N - 2^2;\ 2^N - 2^3;\ \ldots;\ 2^N - 2^{N-1}\right\}. \quad (24)$$

One can write every term of this sequence like:

$$\begin{aligned} &\forall M, N \in \mathbb{N} \wedge M < N : \\ &Q(M,N) = 2^N - 2^M = \\ &2^M\left(2^{N-M} - 1\right) = 2^M M(N-M) \end{aligned} \quad (25)$$

The first term in the sequence is $Q(1,N) = P(N)$ the number defined as a pre-Mersenne number. For example, the sequence in particular for $N=4$ is:

$$\begin{aligned} N = 4 &: \left\{2^4 - 2^1; 2^4 - 2^2; 2^4 - 2^3\right\} \\ &= \{14;12;8\} \end{aligned} \quad (26)$$

The exciting thing is that one can use the numbers constructed as differences of powers of 2 as the pre-Mersenne numbers to obtain paired primes, like in the following particular example:

$$1373 + 2^{43} - 2^{11} = 8796093021533, \quad (27)$$

where a prime number as a source generates a more significant paired prime using a gap: $Q(43,11)$.

In this manner, the whole set of prime numbers might be considered a set of paired primes, including prime numbers separated by gaps like those found in twin primes, extended twin primes, or sums of powers of 2.

4.6. Algorithm to generate primes starting from arbitrary odd numbers

A straightforward algorithm to find extended twin primes can be defined as follows.

The algorithm, an arbitrarily odd number *m* chosen, generates some prime number ahead possessing a gap associated with a power of 2. Within the code, it is supposed a primality test included.

Algorithm 1

```
! Calculation of an extended twin prime gap:
    Enter m;      ! m is odd
    N = 1; Two = 2;
    q ← m + 2;
    while q .not.prime:
      N ← N + 1; Two = Two * 2;
      q ← m + Two;
 ! N builds the 2^N gap between m and prime q
    Print N;q;
```

Calculations of other kinds of paired primes with different sorts of gaps follow this computational scheme, with changing the nature of the gap.

5. Conjectures on Prime Number Generation

Such results permit us to consider a conjecture associated with prime numbers generation.

Conjecture 1:

One can generate every prime number from a previous prime number by adding an element of the $\mathbf{2}^{\mathbb{N}}$ set or a number of the type $Q(M,N)$.

The primes generated by way of *Conjecture 1* can be called *paired primes*.

Another fashion to write the previous conjecture might be:

Conjecture 2.

Every prime number is paired.

One can enunciate this last conjecture because all the even numbers are expressible as sums of the set $\mathbf{2}^{\mathbb{N}}$.

6. Conclusions

The definition of Mersenne intervals easily permits predicting the generation of natural numbers through a recursive algorithm, as described using a vector form in the equation (9).

Such an algorithm precludes that all natural numbers can be generated with a segment of known ordered elements by adding a convenient power of 2. Therefore, the prime numbers cannot be an exception and thus can be supposedly generated by summing powers of 2 or sums of them to odd composite numbers or prime numbers.

Then, one might consider the set of prime numbers as paired primes connected by a power of 2 or a sum of powers of 2.

The prime number set is thus a set of *paired primes* whose gaps might be a power of 2, a sum of powers of 2, or a general pre-Mersenne number.

Twin primes are just a subset of the paired primes set.

One can conjecture an infinite cardinality attached to the paired primes set.

Compliance with ethical standards

Conflict of interest: The authors state that this work has no conflict of interest.

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