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A Generalization of Bernoulli's Inequality

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Abstract: *In this paper we will give a generalization of the celebratedBernoulli inequality proposed by Jacob Bernoulli in the late seventeenth century.*

Keywords: *Bernoulli Inequality.*

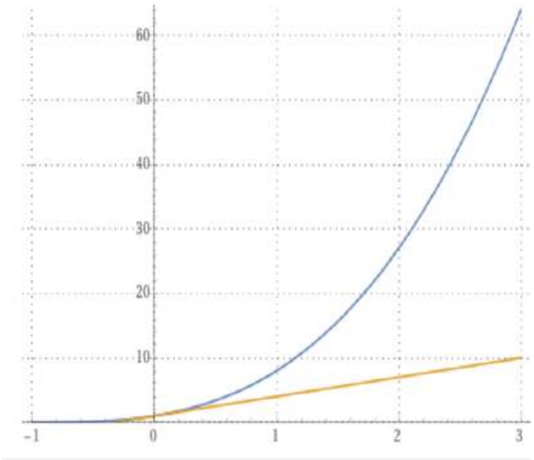
1. Introduction

As is well-known, ***Bernoulli's inequality*** provides a lower bound for $(1+t)^r$. There are several commonly stated forms of this inequality. One of these is

For every integer $r \geq 1$ and all real $t \geq -1$

$$(1+t)^r \geq 1+rt$$

For example, for $r = 3$, the graphs of $f(t) = (1 +$ *t)3 and gt=1+3t* clearly show this to be true:

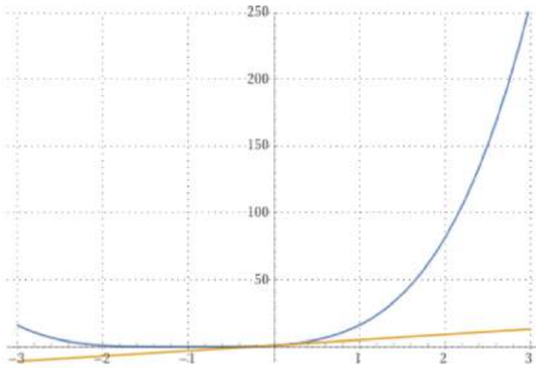


Another common formulation is the following:

For every even integer r and all real t

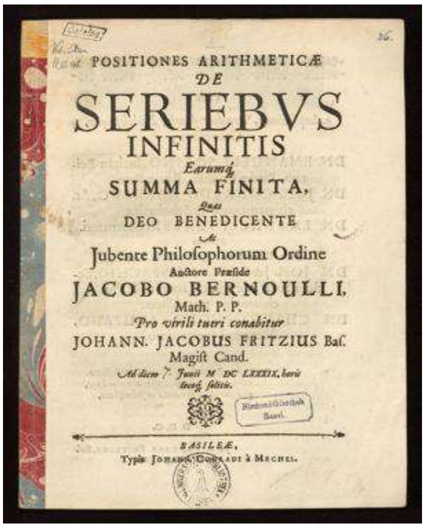
$$(1+t)^r \geq 1+rt$$

This version is depicted for $r = 4$, by the graphs of $f(t) = (1+t)^4$ *and* $g(t) = 1+4t$



For more information see Bullen, P. S. (2003).

The inequality first appeared in Jacob Bernoulli's treatise ***Positiones Arithmeticae de Seriebus Infinitis*** (Basel, 1689).



POSITIONES ARITHMETICÆ
DE
SERIEBVS
INFINITIS
Earumq;
SUMMA FINITA,
Quas
DEO BENEDICENTE
Et
Jubente Philosophorum Ordine
Auctore Præside
JACOBO BERNOULLI,
Math. P. P.
Pro virili tueri conabitur
JOHANN. JACOBUS FRITZIUS Bas.
Magist. Cand.
Ad diem 7. Junii M DC LXXXIX, horis locoq; solitis.
BASILEÆ,
Typis Johann. Conradi à Mechel.

However, it is claimed that the inequality is actually due to French polymath René de Sluze. In 1659, de Sluze published his famous ***Mesolabum seu duse mediae proportionales inter extremas datas per circulum et ellipsim vel hyperbolam infinitis modis exhibitae***[1]. In this work, Sluze, among other things, presented the solutions ofthe third- and fourth-degree equations which he obtained geometrically using the intersection of any conic section with a circle and used this inequality. Indeed, the family of curves

[1] Work on two mean proportional between the two measurements given by a circle and an ellipse or an infinite hyperbola.

$y^n = k(a-x)^p x^m$ for n, p, m positive integers are still called the ***pearls of Sluze***. For details, see Hofmann (1964).

2. The Main Theorem

Let k, mbe positive integers with $1 \le m \le k-1$. We define the function $S(t,k,m)$as

$$S(t,k,m) = t^k - \binom{k}{0} - \binom{k}{1}(t-1) - \binom{k}{2}(t-1)^2 - \cdots - \binom{k}{m}(t-1)^m$$

Now let us proceed to prove our main theorem.

Theorem 1. *There exists a function*$U(t,k,m)$ *such that*

$$S(t,k,m) = (t-1)^{m+1} U(t,k,m)$$

and

$$U(1,k,m) = \binom{k}{m+1}$$

Proof. The proof is by induction on m. Let us show that the result holds for $m = 1$.

$$S(t,k,1) = (t-1)(t^{k-1} + t^{k-2} + \cdots + 1 - k)$$

$$= (t-1)[(t^{k-1} - 1) + (t^{k-2} - 1) + \cdots + (t^2 - 1) + (t-1)]$$

$$= (t-1)^2[(t^{k-2} + \cdots + 1) + (t^{k-3} + \cdots + 1) + \cdots + (t+1) + 1]$$

Let

$$U(t,k,1) = [(t^{k-2} + \cdots + 1) + (t^{k-3} + \cdots + 1) + \cdots + (t+1) + 1]$$

Then

$$S(t,k,1) = (t-1)^2 U(t,k,1)$$

and

$$U(1,k,1) = (k-1) + (k-2) + \cdots + 1 = \frac{k(k-1)}{2} = \binom{k}{2}$$

Now let us suppose the theorem is true for m and let us prove it to be true for $m+1$:

$$S(t,k,m+1) = t^k - \binom{k}{0} - \binom{k}{1}(t-1) - \binom{k}{2}(t-1)^2 - \cdots - \binom{k}{m}(t-1)^m - \binom{k}{m+1}(t-1)^{m+1}$$

$$= (t-1)^{m+1} U(t,k,m) - \binom{k}{m+1}(t-1)^{m+1}$$

$$= (t-1)^{m+1}\left[U(t,k,m) - \binom{k}{m+1}\right]$$

Since

$$U(t,k,m) - \binom{k}{m+1}$$

$$= (t^{k-m+1} + \cdots + 1) + (t^{k-m-1} + \cdots + 1) + \cdots + (t+1)$$

$$= (t-1)U(t,k,m+1)$$

the result follows.

We claim that Theorem 1 is actually a generalization of the Bernoulli's Inequality

Indeed,

$$S(t,k,1) = t^k - 1 - k(t-1)$$

Thus,

For $t \ge 0$, $U(t,k,1)$ and thus $S(t,k,1) \ge 0$, which is Bernoulli's inequality. More specifically,

Theorem 2 (Generalized Bernoulli's Inequality)

(i) *If*m*is an odd natural number and*$1 \le m \le k-1$, *we have*

$$S(t,k,m) \ge 0$$

Equality holds only if$t = 1$.

(ii) *If*$t \ge 1$*then for any*$1 \le m \le k-1$, *we have*

$$S(t,k,m) \ge 0$$

Equality holds only if$t = 1$.

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Hofmann, Joseph E. (1964) Über die Exercitatio Geometrica des M. A. Ricci. ***Centaurus***, Vol. 9, Issue 3, p.139-193, March 1964.

