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On the first passage time problem for dynamic systems

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Summary: *The first-time passage problem of a dynamic system is of paramount importance to check the system reliability. In this paper the problem is analyzed in the case of Fokker-Planck Markov process responses. Both the differential approach of solution and the integral one are reviewed, but the latter is followed in the applications. These regard the Ornstein-Uhlenbeck process and the envelope of the response of an oscillator with nonlinear stiffness.*

Keywords: *dynamic systems, scalar Markov processes, Fokker-Planck equation, first-passage time statistics.*

1. Introduction

It is often required that an engineering dynamic system vibrates within safe prescribed limits in a given time interval. If the system is acted by stochastic loads, the problem must be formulated in a probabilistic context: the probability that the system stay in these limits must be evaluated. This is a very important problem in the field of stochastic process theory and applications; it is called first-passage time problem that arises in physics, chemistry, engineering and biology. In this problem, the time when a stochastic process first reaches a threshold value, the barrier, is studied.

Be $X(t)$ a stochastic process defined for continuous time t. Let $X_M(t)$ the largest value of X in a time interval [0, T]: $X_M(t) = \max \{X(t),\ 0 \le t \le T\}$. If T_P is the time when X first crosses a barrier, clearly the event $X_M(t) \le b$ implies $T_P \ge T$. Thus, the following probabilities are equal:

$P[X_M \le b | [0,T]] = P[T_P > T]$. In other words $P[X_M(t) \le b | [0,T]] = P[T_P \ge T] = 1 - P[T_P < T]$.

The two problems so defined are complementary, and one can looks for either probabilities.

Unfortunately, exact analytical solutions for the first-time passage statistics exist only in the case of scalar systems, and these suffer from the drawback of being cumbersome. In the case of second and higher order systems, no exact solutions are available, and the analyst must recur to approximate methods. On the other hand, the results obtainable for scalar systems are useful for systems for two or more states. In fact, by referring to the envelope for linear systems, and by applying the principles of stochastic averaging for nonlinear ones, the analysis of the response of second order and higher order systems is reduced to the analysis of a scalar stochastic Markovian process.

Probably, the study of the first-passage time problem was initiated by Rice with his inclusion-exclusion series [1]. By now, the literature on the argument is very numerous [2 - 44]: the list of the references is not complete, and in some way it reflects the personal perspective of the writer. In [2 - 7, 9, 10, 12, 14, 17, 19, 20, 23, 27, 30, 31, 33, 35, 37, 38, 42] the first passage time is studied for scalar Markovian processes, which in most cases are Gaussian. The majority of the authors uses a differential approach, but in [30, 33, 35, 38] Volterra integral equations are derived, whose solution is the probability density function (PDF) of the first-passage time. Analytical solutions are obtained in particular for Wiener and Ornstein-Uhlenbeck (OU) processes. The drawbacks inherent to these solutions are in that or they are expressed as infinite series of trascendental functions or as Laplace transforms that are difficult to invert. Thus, in the past the analytical solutions have not been completely exploited, and asymptotic estimates have been searched.

Many authors have studied the first-passage problem for second order oscillators: [11, 13, 15, 16, 18, 19, 21, 24, 25, 28, 29, 32, 34, 36, 37, 40, 41, 42]. On the contrary, there are few studies on higher order oscillators: see [39, 43, 44]. Some authors have obtained even fair results by using both analytical methods and semi-empirical considerations: [19, 21, 25, 36, 40, 41]. In many studies the problem is reduced to the analysis of a scalar process: [8, 11, 16, 18, 21, 22, 24, 28, 29, 34, 36]. This is accomplished by considering the envelope of the response [45] for linear oscillators, while for nonlinear ones the stochastic averaging method [46] is used. In [13, 15, 37, 42] Rice's inclusion-exclusion series is exploited, while Madsen and Krenk [32] derive an integral equation.

This paper is concerned with the first-passage problem of a scalar Markov process described by the Fokker-Planck-Kolmogorov (FPK) equation [47]. This choice is not restrictive inasmuch as the study of response of second order oscillators is reducible to this case: because of lack of space this topics is not discussed (e.g. see [29, 34]). In Sec. 2 the FPK equation with initial and boundary conditions suitable for this problem is established. Then, the expression of the first-passage time PDF is obtained. Secs. 3 and 4 are devoted to the applications and to the conclusions, respectively.

2. Problem Statement

Consider the stochastic process X: $X(-\infty, +\infty) \times \Omega \to \Re$, defined on a probability space (Ω, Σ, P) be an Itô diffusion process satisfying a stochastic differential equation of the form

$$\mathrm{d}X(t) = a(X(t),t)\,\mathrm{d}t + b(X(t),t)\,\mathrm{d}B \quad \left(X(t_0) = x_0\right), \qquad (1)$$

where $B(t)$ is a scalar Wiener process (Brownian motion), $a : \Re \to \Re$ and $b : \Re \to \Re$ are the drift and diffusion coefficients, respectively. Eq. (1) is a generalized Langevin equation. Thus, the infinitesimal generator of X is the Fokker-Planck one, that is:

$$\frac{\partial p\left(x,t|x_0,t_0\right)}{\partial t} = -\frac{\partial}{\partial x}\left[m(x)\,p\left(x,t|x_0,t_0\right)\right] + \frac{1}{2}\frac{\partial}{\partial x}\left[b^2(x)\frac{\partial p\left(x,t|x_0,t_0\right)}{\partial x}\right], \qquad (2)$$

where $p\left(x,t|x_0,t_0\right)$ is the transition probability density function (PDF) of X given $X(t_0) = x_0$. In order to find $p\left(x,t|x_0,t_0\right)$, the initial and boundary conditions for Eq. (2) are $p\left(x,t_0|x_0,t_0\right) = \delta(x - x_0)$ and $p\left(x,t|x_0,t_0\right) \downarrow 0$ as $|x| \uparrow \infty$. The relationship between $a(X)$ in Eq. (1) and $m(x)$ in Eq. 2 is $m(x) = a(x) + \frac{1}{2}b(x)\frac{\mathrm{d}b}{\mathrm{d}x}$, in which the second term is the so-called Wong-Zakai-Stratonovich corrective term (WZ) [48, 49]. This term is zero if the diffusion coefficient does not depend on X. The problem of the first-passage time of the process $X(t)$ so defined can be solved in two ways: the differential approach, and the integral approach, whose fundamental principles are given below.

2.1 Differential approach

Let $f\left(x,t|x_0\right)\mathrm{d}x$ be the probability that $X(t)$ takes a value within $(x,\, x + \mathrm{d}x)$ without having crossed $x = b$ in the time $(0, T)$, given $X(0) = x_0$. It can be shown that $f\left(x,t|x_0\right)$ satisfies Eq. (2) with the same initial condition and different boundary conditions. In the barrier b an absorbing boundary is placed, which requires $f\left(b,t|x_0\right) = 0$. Since another boundary condition is necessary, a reflecting boundary is placed at $x = r$, which requires [14, 20]

$$\left\{-m(x)\,f\left(x,t|x_0\right) + \frac{1}{2}b^2(x)\frac{\partial f\left(x,t|x_0\right)}{\partial x}\right\}_{x=r} = 0. \qquad (3)$$

The solution of the FPK equation (2) in the unknown PDF $f\left(x,t|x_0\right)$ with the initial and boundary conditions so established can be looked for either by means of the Laplace transform method (1) or by separation of variables. In the latter case the solution can be put in the equivalent forms

$$f\left(x,t|x_0\right) = \rho(x)\sum_{0}^{\infty}{}_k\, c_k\,\phi_k(x)\,\phi_k(x_0)\,e^{-\lambda_k t} = \sum_{0}^{+\infty}{}_k\, a_k\,\phi_k(x_0,\lambda_k)\,e^{-\lambda_k t}. \qquad (4)$$

In the intermediate member of Eq. (4) $\rho(x)$ is the steady-state PDF of $X(t)$, that is the solution of Eq. (2) with the right-hand-side zero, that is

$$\rho(x) = \frac{c}{b^2(x)}\exp\left[\int_0^x \frac{m(u)}{b^2(u)}\mathrm{d}u\right], \qquad (5)$$

where c is a normalization constant. The constants c_k are expressed as

$$c_k\delta_{km} = \left(\int_\eta^\xi \rho(x)\phi_k(x)\phi_m(x)\,\mathrm{d}x\right)^{-1}, \qquad (6)$$

where δ is Krönecker symbol, while η and ξ depend on the boundary conditions (see [20]). The constants a_k are given by

$$a_k = \frac{\int_\xi^\eta \phi_k(y,\lambda_k)\,\mathrm{d}y}{\int_\xi^\eta \phi_k^2(y,\lambda_k)\,\mathrm{d}y}. \qquad (7)$$

Even in this case the limits of integration depend on the boundary conditions: see [50].

The eigen functions $\phi_k(x)$ and the eigenvalues λ_k satisfy the following Sturm-Liouville problem:

$$b^2(x)\phi_k'' + 2m(x)\phi_k'(x) + 2\lambda_k\phi_k(x) = 0, \qquad (8)$$

where the apexes mean derivative with respect to x. Eq. (8) has the boundary conditions

$$\rho(r)\phi_k'(r) = 0 \qquad \rho(b)\phi_k(b) = 0, \qquad (9)$$

for the case of one reflecting boundary at $x = r$ and one absorbing boundary at $x = b$, the so-called B-type barrier. For this type of barrier Eq. (6) becomes

$$\delta_{km} c_k^{-1} = \int_{\xi}^{\eta} \rho(x)\, \phi_k(x)\, \phi_m(x)\, \mathrm{d}x\,. \tag{10}$$

The use of the last member of Eq. (4) is analogous: see [50].

Once the PDF $f\left(x,t\,|\,x_0\right)$ has been computed, the cumulative distribution function (CDF) of T_P is obtained as

$$F_{T_P}\left(t,b\,|\,x_0\right) = P\left[T_P(b) \le t\,|\,X(0) = x_0\right] =$$

$$1 - \int_{\xi}^{\eta} f\left(x,t\,|\,x_0\right)\mathrm{d}x\,. \tag{11}$$

It must be pointed out that $\int_r^b f\left(x,t\,|\,x_0\right)\mathrm{d}x = F_{X_M}\left(b,t\,|\,x_0\right)$, which is the CDF of the largest value X_M of X.

Some authors proposed asymptotic approximations valid for high barrier and/or long t. One is [6, 20, 21, 27]

$$F_{T_P}\left(t,b\,|\,x_0\right) \approx 1 - e^{-\lambda_0(b)\,t}\,. \tag{12}$$

Eq. (12) looks like Cramer's asymptotic CDF of a Gaussian process [51], which is based on the hypothesis of independent excursions: the mean upcrossing rate of zero line, which is infinite for a first order Markov process, is replaced by the first eigenvalue λ_0. Another asympotic CDF has the form of a Gumbel distribution:

$$F_{T_P}\left(t,b\,|\,x_0\right) \to \exp\{-\exp[-(x+\alpha)/\beta]\}$$

(the expressions of α and β are given in [6]). The statistical moments of the first-passage time are

$$E\left[T_P^n\right] = n! \sum_0^{\infty} {}_k \frac{B_k(b)}{\lambda_k(b)} \to \frac{n!}{\lambda_0^n(b)}\,, \tag{13}$$

where $B_k(b) = c_k \left[\int_r^b \rho(x)\,\phi_k(x)\,\mathrm{d}x\right]^2$.

Different but equivalent expressions can be found elsewhere, e.g. in [11, 18]. In the limit the mean first passage time is $1/\lambda_0$, which has a Poissonian fashion. It is recalled that the statistical moments of the first passage time can be computed by means of the Andronov-Pontriagin-Vitt equation that is no treated here for brevity's sake: see [52]. However, only the equation for the first moment is in general affordable.

2.2 Integral approach

Let $p_T\left(b\,|\,x_0,t\right)$ be the first passage probability density for $x = b$: thus, $p_T\left(b\,|\,x_0,t\right)\mathrm{d}t$ is the probability that $x = b$ is attained the first time in $[t, t+\mathrm{d}t)$ given the initial condition. Since $X(t)$ is a Markov process, the Chapman-Kolmogorov-Smoluchovski equations holds, which is written for the PDF's $p_T\left(b\,|\,x_0,t\right)$ and $f\left(x,t\,|\,x_0\right)$ [2, 3]:

$$f\left(x,t\,|\,x_0\right) = \int_0^t p_T\left(b\,|\,x_0,\tau\right)\cdot f\left(x,t-\tau\,|\,b\right)\mathrm{d}\tau\,.$$

Eq. (12) is a convolution integral: taking the Laplace transform of both sides and applying the con-volution theorem one obtains

$$\mathsf{P}\left(b\,|\,x_0,s\right) = \frac{\mathsf{F}\left(x,s\,|\,x_0\right)}{\mathsf{F}\left(b,s\,|\,x_0\right)}, \tag{13}$$

where s is the Laplace transform parameter, $\mathsf{P}\left(b\,|\,x_0,s\right)$ is the transform of $p_T\left(b\,|\,x_0,t\right)$, while $\mathsf{F}\left(x,s\,|\,x_0\right)$ is that of $f\left(x,t\,|\,x_0\right)$. Darling and Siegert was not able to invert the Laplace transform in any case [2, 3], what was accomplished by Gray in some cases [12]. Moreover, he gave the general formula for the first-passage time CDF

$$Q_T(b,t) = 1 - \sum_0^{\infty} {}_k \frac{\mathsf{F}\left(x,-\alpha_k\,|\,x_0\right)}{\alpha_k\, \mathsf{F}'\left(b,-\alpha_k\,|\,x_0\right)} e^{-\alpha_k t}\,, \tag{14}$$

where $-\alpha_k$ are the zeros of $\mathsf{F}\left(b,s\,|\,x_0\right)$, and the apex means derivative with respect to s. The α_k's are real numbers. The first-passage time CDF is denoted with the different symbol $Q_T(b,t)$ to underline that it is computed by means of a different approach.

The first-passage time moments $M_n^T(b) = E\left[T_P^n(b)\right]$ can be obtained directly from the result (13) [33]:

$$M_n^T(b) = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}\, \mathsf{P}\left(b\,|\,x_0,s\right)_{s=0}\,. \tag{15}$$

3 Applications

The applications concern the OU process solution to the classic Langevin equation, the energy envelope of a second order oscillator with nonlinear restoring force having the form $r(x) = k\,|x|^n\,\mathrm{sgn}(x)$, and the second order linear oscillator.

3.1 First passage of the OU process

The OU process is the solution of the stochastic differential equation (Langevin equation)

$$\mathrm{d}X(t) = -\mu X(t)\,\mathrm{d}t + \sqrt{2K}\,\mathrm{d}B \quad \left(X(t_0) = x_0\right), \tag{16}$$

where $B(t)$ is a unit strength Wiener process. The corresponding FPK equation is

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x}(\mu\, x\, f\,) + K\frac{\partial^2 f}{\partial x^2}. \tag{17}$$

In the equilibrium regime the PDF holds the form (4) with

$$\rho(x) = \frac{1}{\sqrt{2\pi(K/\mu)}}\exp\left(-\frac{\mu x^2}{2K}\right), \tag{18}$$

which is a Gaussian PDF.

Before proceeding, the following dimensionless variables are introduced: $y = \sqrt{\mu/2K}\,x$, $\tau = \mu t$; the barrier becomes $\xi = \sqrt{\mu/2K}\,b$. The Sturm-Liouville problem is rewritten as

$$\phi_k''(y) - 2y\,\phi_k'(y) + 2\alpha_k \phi_k(y) = 0, \tag{19}$$

where the apexes mean derivatives with respect to *y*. An absorbing boundary condition is placed at y = ξ, while a reflecting boundary condition is placed at *y* = *r*. For the brevity's sake, the derivation of the eigenfunctions is not reported; however, it is emphasized that these are dependent on the value of *r*. The simplest case is for *r* = 0 when the eigenfunctions are the confluent hypergeometric

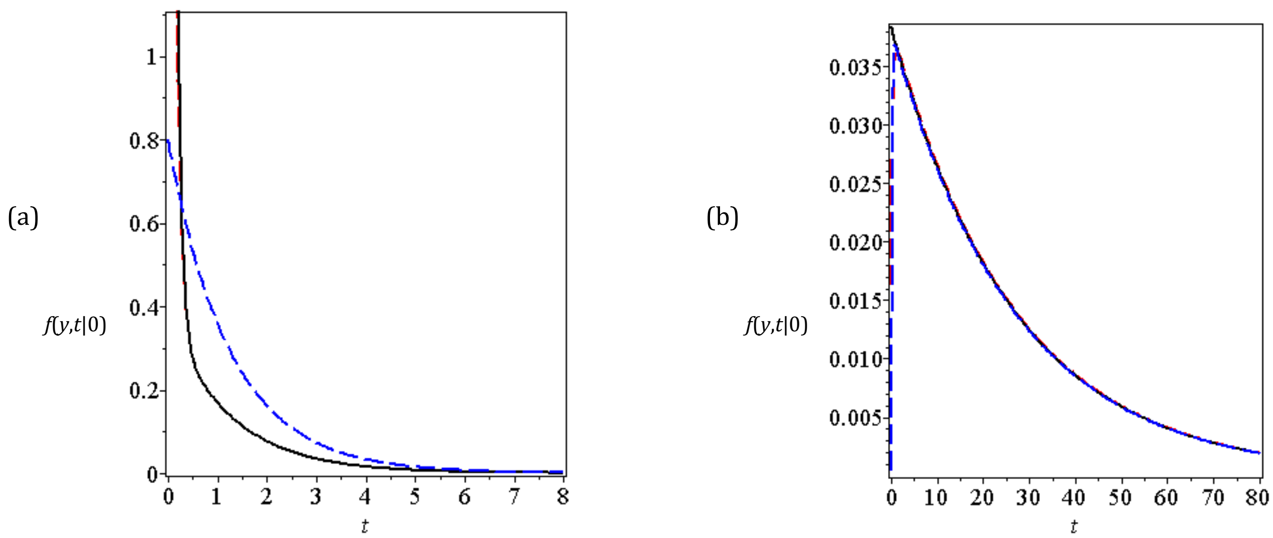

Figure 1: comparison of the PDF's $f(y,t|0)$ for $\xi = 1$ (a), and $\xi = 4$ (b). Blue line: one term in the series; red line two terms; black line three terms.

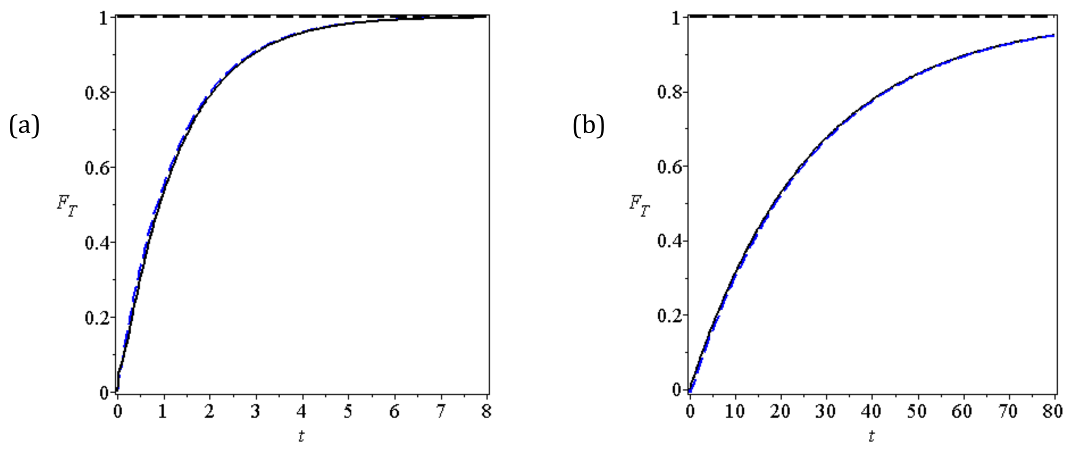

Figure 2: comparison of the CDF's of the first-passage time of the OU process: ξ = 1 (a), and ξ = 4 (b). Blue line: Eq. (10); black line Eq. (14) with three terms.

functions [53]. Hence, the eigenvalues are the roots of the trascendental equations

${}_1F_1\left(-\alpha_k/2,\,1/2,\xi^2\right)=0$: they are all real and positive numbers. Taking the Laplace transform of $f\left(x,t\middle|x_0\right)$ one obtains

$$\mathrm{P}\left(b\middle|x_0,s\right)=\frac{\sum\limits_{k=0\ldots+\infty} c_k\,\phi_k(x)\,\phi_k(y_0)\left(s+\alpha_k\right)^{-1}}{\sum\limits_{k=0\ldots+\infty} c_k\,\phi_k(\xi)\,\phi_k(y_0)\left(s+\alpha_k\right)^{-1}}.\qquad(20)$$

It is easily recognized that the poles of the denominator are the eigenvalues α_k: one returns to the time domain, and uses the formula (14).

The parameters take the following values: $K = 1$, $\mu = 1$. In this way, $E[X^2]=\sigma_X^2=K/\mu=1$. The dimensionless barrier is $\xi=\sqrt{\mu/2K}\,b$. The initial condition is $X(0) = 0$. When $\xi = 1$ the first three eigenvalues of the confluent hypergeometric function ${}_1F_1\left(-\alpha_k/2,\,1/2,\xi^2\right)$ are: 0.798460, 10.758826, 30.506530. When $\xi = 4$ $\lambda_0=0.037517$, $\lambda_1=2.899944$, $\lambda_2=7.868869$. Clearly, the first eigenvalue is always dominant. In Fig . 1 the transition PDF's $f\left(y,t\middle|0\right)$ solutions of the FPK equation (17) are plotted versus the non-dimensional time, which is still denoted by t in the plots. The left plot refers to $\xi = 1$, while the right plot to $\xi = 4$. In the case of low barrier $\xi = 1$ one term in the series is not sufficient to have an acceptable result, but two term are sufficient: adding a third term there are no appreciable changes. In the case of high barrier $\xi = 4$ one term in the series gives good results with the exception of the first two units of time. The curves with the two terms and with three terms are near superimposed: there is a very steep increase for very short values of t; then, they have a slow decay.

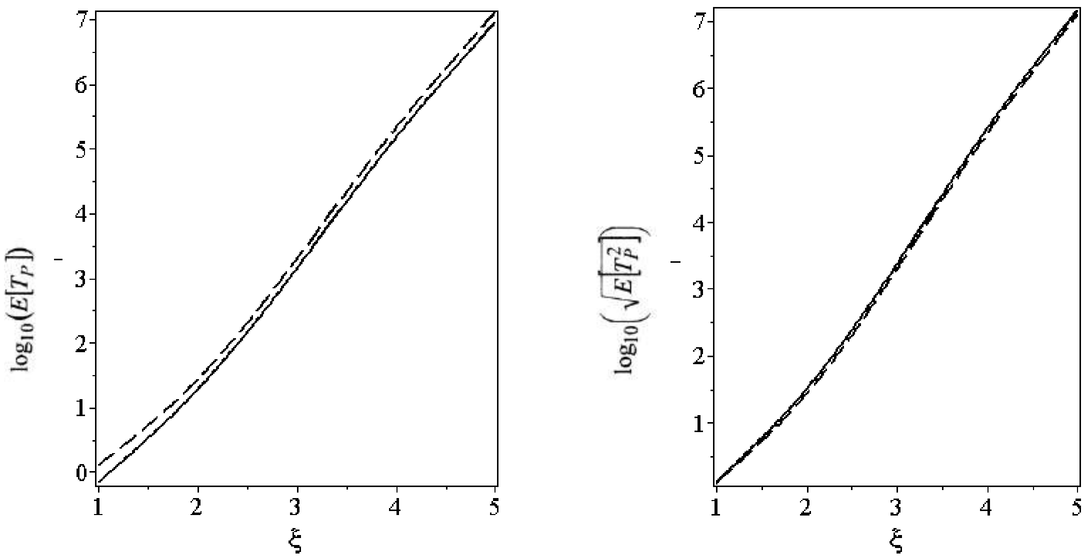

Figure 3: left $\log_{10}\left(E\left[T_P\right]\right)$, right $\log_{10}\left[\left(E\left[T_P^2\right]\right)^{1/2}\right]$ against the barrier ξ; Eq. (11) solid line, asymptotic sum of the series in Eq. (11) dashed line

In Fig. 2 there are the first-passage time CDF's for $\xi = 1$ (a), and $\xi = 4$ (b), respectively. The estimates of the series (14) are compared with the asymptotic approximation (10): the agreement is almost perfect with the exception of a very short time interval starting from zero in the plot (a), which cannot be perceived examining the figure.

In Fig. 3 there are the curves of the mean $E\left[T_P\right]$ and of the mean square $E\left[T_P^2\right]$ of the first-passage time with respect the barrier level ξ. As both quantities take large values as ξ increases, in the ordinates there are the logarithms $\log_{10}\left(E\left[T_P\right]\right)$ and $\log_{10}\left[\left(E\left[T_P^2\right]\right)^{1/2}\right]$, respectively. The series in Eq. (11) are evaluated by taking 4 or 5 terms into account; however, the terms beyond the second give a little contribution. The curves deriving from the series are the continuous line in the plots, while the dashed lines are obtained by means of the asymptotic sum of the series as in the right-hand-side of Eq. (11). There is a substantial agreement between the two approaches. The general trends and the large values of the moments agree with the findings of [20] for different boundary conditions.

3.2 Oscillator with nonlinear restoring force
Consider the oscillator

$$\ddot{X} + \beta\dot{X} + \omega_0^2 |X|^n \operatorname{sgn}(X) = \sqrt{2\pi w_0} W(t), \quad (21)$$

where $\operatorname{sgn}(\bullet)$ denotes the signum function, and $W(t)$ is a Gaussian white noise with unit power

spectral density. The mechanical energy of the oscillator is

$$E(x,\dot{x}) = \frac{\dot{x}^2}{2} + \frac{\omega_0^2}{n+1}|x|^{n+1}. \quad (22)$$

For brevity's sake the derivation of the stochastic differential equation for the energy E is not detailed: see [24, 46, 54]. It reads as

$$\mathrm{d}E = m(E)\,\mathrm{d}t + \sigma(E)\,\mathrm{d}B(t), \quad (23)$$

which is analogous to the Langevin's equation. In Eq. (23) $B(t)$ is the standard Brownian motion, the formal derivative of the unit strength Gaussian white noise $W(t)$. In the case of only one external excitation of Gaussian white noise type the drift and the diffusion coefficients are, respectively

$$m(E) = -\beta\left\langle \dot{X}\, h(X,\dot{X})\right\rangle_t + \pi w_0$$
$$\sigma(E) = 2\beta\pi w_0 \left\langle \ddot{X}^2 \right\rangle = 4\beta\pi w_0 \frac{n+1}{n+3} E \quad (24\text{ a,b}),$$

where $h(X,\dot{X})$ is the damping function (simply $h(X,\dot{X}) = \dot{X}$ in this case) and $\langle \bullet \rangle_t$ the time averaging operator for which the reader is referred to [24, 46, 54].

At this point one proceeds as in the scalar case. Omitting the passages, the CDF of the first passage time is given by

$$F_T(t,\xi) = 1 - \sum_0^\infty k\, c_k\; {}_1F_1\left(-\frac{\alpha_k}{a}, \frac{1}{a}, \frac{\xi}{c}\right) e^{-\alpha_k \beta t}. \quad (25)$$

Again, the eigenvalues α_k are those of the confluent hypergeometric function; ξ is the barrier for the non-dimensional energy $\Lambda = E/\left(\omega_0^2\, \sigma_E^{n+1}\right)$. The expressions of the coefficients c_k as well as those of the constants a and c are not reported because for brevity's sake; the latter depend on the system parameters only. It is emphasized that this approach is related to the crossings of a critical value of the energy (type E barrier). On the contrary, in the case of Eqs. (1) and (16) the crossings of the barrier $X(t) = b$ (type B barrier) or of the double barrier $X(t) = \pm b$ (type D barrier) are considered.

There is not a relationship among the statistical quantities of the E crossings and those of B and D crossings.

For the same reason only the first-passage time CDF's of the cases $\beta = 1$, $n = 3$, $\omega_0 = 2\pi$, $\xi = 2, 4$ are shown in Fig. 4. In this case using one term in the series does not give a realistic picture of the CDF. However, three terms are sufficient. The statistical averages of the first passage time are 0.324 s and 0.886 s for the two non-dimensional barriers, respectively,

3.3 Second order linear oscillator

If in eq. (21) $n = 1$, we retrieve the second order linear oscillator, say

$$\ddot{X} + \beta\dot{X} + \omega_0^2 X = \sqrt{2\pi w_0}\, W(t). \quad (26)$$

For a second order linear oscillator several approaches have been proposed for studying the prob-lem of the first passage time: however, Eqs. (23 -25) are valid even in this case and are used herein. The first-passage time CDF's for $\beta = 1$, $n = 1$,
$\omega_0 = 2\pi$, $w_0 = 1.0$, $\xi = 2, 4$ are shown in Fig. 5.

One term in the series (25) is not sufficient to seize the exact CDF, but three terms are for the low barrier; four terms are necessary for the highest barrier. It is noted that the CDF's are very steep for

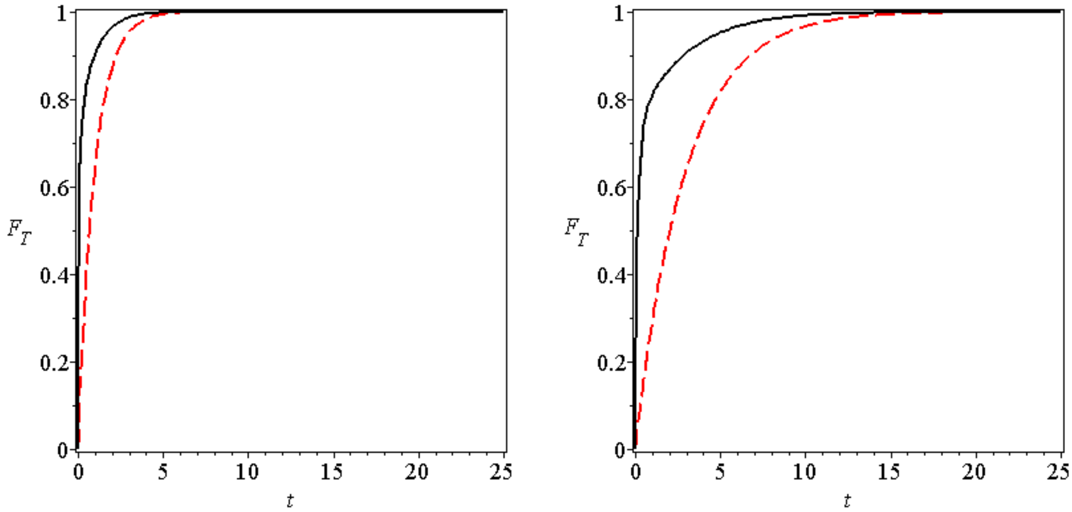

Fig. 4: CDF of the first-passage time for the energy of the oscillator (21) with $\beta = 1$, $n = 3$, $\omega_0 = 2\pi$, $\xi = 2$ in the left plot, $\xi = 4$ in the right plot; red line one term in the series (25), black line three terms.

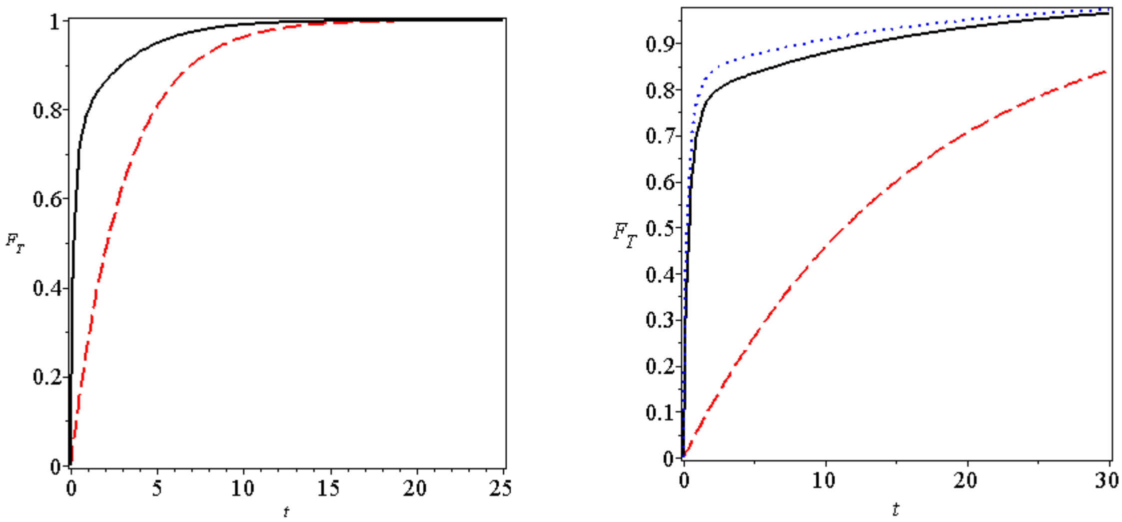

Fig. 5: CDF of the first-passage time for the energy of the linear oscillator (26) with $\beta = 1$, $n = 1$, $\omega_0 = 2\pi$, $\xi = 2$ in the left plot, $\xi = 4$ in the right plot; red line one term in the series (25), black line three terms, blue dotted line four terms.

short times; then, they become smooth. For the highest barrier including only one term in the series causes very gross errors. The average first passage times are 0.952 s and 3.019 s for the two barriers respectively.

As previously advanced, even in the simplest case of the linear second order oscillator there are not exact results for the first passage time problem, but many approximate methods were proposed. As the CDF's in Fig. 5 refers to an E type barrier, now we want to consider the D barrier problem. The simplest approximation is to assume that upcrossings and downcrossings constitute a homogeneous Poisson process [51]. As the response of the oscillator (26) is Gaussian, the CDF of the first passage time for a type D barrier is given by

$$F_T(t,\xi) = 1 - \exp\left(-2\nu_0 t\, e^{-\xi^2/2}\right), \tag{27}$$

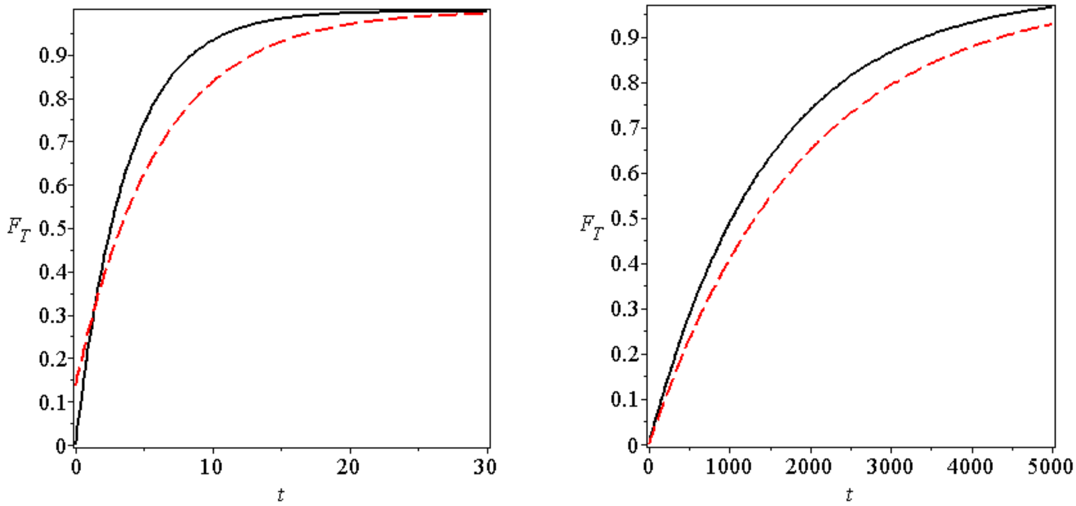

Fig. 6: CDF of the first-passage time for the displacement of the linear oscillator (26) with β = 1, n = 1, $\omega_0 = 2\pi$, $\xi = 2$ in the left plot, $\xi = 4$ in the right plot; red line Vanmarcke's equation (29), black line Poisson assumption (27).

where $\xi = b/\sigma_X$ (here b is a displacement), and ν_0 is the mean upcrossing rate of zero line, say

$$\nu_0 = \frac{\sigma_{\dot{X}}}{2\pi\sigma_X}. \tag{28}$$

As in this case

$$\sigma_X^2 = \pi w_0/\left(2\beta\omega_0^2\right),\ \sigma_{\dot{X}}^2 = \pi w_0/2\beta, \text{ Eq. } (28)$$

reduces simply to $\omega_0/2\pi$.

The results of Eq. (27) are compared with the CDF proposed by Vanmarcke [19], which in part stems from empirical considerations:

$$F_T(t,\xi) = 1-\left[1-\exp\left(-\frac{\xi^2}{2}\right)\right]\cdot\exp\left\{-2\nu_0 t\, e^{-\xi^2/2}\cdot\frac{\left[1-\exp\left(-\sqrt{\pi/2}\right)q\xi\right]}{1-\exp\left(-\xi^2/2\right)}\right\}. \tag{29}$$

In Eq. (29) q is the so-called Vanmarcke's parameter: $q = \sqrt{1-\lambda_1^2/\lambda_0\lambda_2}$, being λ_k $(k = 0,1,2)$ the spectral moments of the one-sided power spectral density of the response of the oscillator.

The CDF's resulting from the approaches are plotted in Fig. 6 for barriers $\xi = 2$ (left plot) and $\xi = 4$ (right plot). Some differences are evidenced as Poisson CDF tends to one faster, but in substance the two approaches are not in contradiction. The differences are more marked in terms of averages of the first passage time: assuming Poisson crossings they are 3.695 s and 1490.48 s for the two barriers, respectively. By using Vanmarcke's Eq. (29) they are 5.177 s and 1841.33 s. In any case the assumption of Poisson crossings is conservative. A direct comparison of the distribution functions of Fig. 6 with those of Fig. 5 is not possible as they refer to two different barriers. However, it is noticeable that crossing a displacement threshold requires much longer times than cross-ing a critical value of the energy.

4 Conclusions

We have examined the first passage problem for scalar Markov processes generated by a generalized Langevin equation. In order to solve the problem, two approaches are possible: the integral approach and the differential approach; both give raise to exact results. On the other hand, in general the integral approach requires working in the Laplace transform domain: in many case the solution that one finds cannot be inverted analytically.

The differential approach is based on writing and solving the Fokker-Planck-Kolmogorov equation or its adjoint the backward Kolmogorov equation with appropriate initial and boundary conditions. Among the methods for solving these equations, in many cases the ansatz of separation of the variables is fruitful: in the next step an eigen problem must be solved.

Unfortunately, exact solutions for the first passage problem do not exist for second and higher order systems. Thus, the analyst must reduce the problem to a scalar one: the stochastic averaging methods are suitable to do this.

The applications regard an Ornstein-Uhlenbeck process and a second order oscillator with restoring force of the form $\omega_0^2 |X|^n \operatorname{sgn}(X)$. In order to reduce the problem to a scalar one, the stochastic averaging of the energy envelope is used: in this way, the crossings of a critical value of the energy is considered. The cases $n=3$ and $n=1$ are examined: in the latter case we have the classical second order oscillator.

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